METASTABLE BOUND STATES OF TWO-DIMENSIONAL MAGNETOEXCITONS IN THE LOWEST LANDAU LEVELS APPROXIMATION

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Abstract

The possible existence of the two-dimensional bimagnetoexcitons and metastable bound states formed by two magnetoexcitons with opposite in-plane wave vectors $\vec{k}$ and $-\vec{k}$ has been studied. Magnetoexcitons taking part in the formation of molecules look as two electric dipoles with the arms oriented in-plane perpendicular to the respective wave vectors and with the length of the arms $d = kl_0^2$, where l_0 is the magnetic length. Two antiparallel dipoles moving with equal, yet antiparallel, wave vectors have the possibility of moving with equal probability in any direction of the plane, which is determined by the trial wave function of relative motion $\varphi_n(|\vec{k}|)$ depending on modulus k. The magnetoexcitons are composed of electrons and holes situated on the lowest Landau levels with the cyclotron energies greater than the binding energy of the 2D Wannier–Mott exciton. The description has been made in Landau gauge. The spin states of two electrons have been chosen in the form of antisymmetric or symmetric combinations with parameter $\eta = \pm 1$. The effective spins of two heavy holes have been combined in the same resultant spinor states as the spin of the electrons. Because the projections of the both spinor states with $\eta = \pm 1$ are equal to zero, the influence of the Zeeman splitting effect vanishes. In the case of trial wave function $\varphi_2(k) = (8\alpha^3)^{1/2} k^2 l_0^2 e^{-\alpha k^2 l_0^2}$, the maximal density of the magnetoexcitons in the momentum space is concentrated on the in-plane ring with radius $k_r = 1/(\sqrt{\alpha} l_0)$. In the approximation of the lowest Landau levels, when the influence of the excited Landau levels is neglected, stable bound states of bimagnetoexcitons do not exist for both spin orientations. Instead, in the case of $\alpha = 0.5$ and $\eta = 1$, a deep metastable bound state with the activation barrier comparable with two magnetoexciton ionization potentials $2I_l$ has been revealed. In the case of $\eta = -1$ and $\alpha = 3.4$, only a shallow metastable bound state can appear.

1. Introduction

The study of the biexciton formation in the presence of an external magnetic field was initiated in [1, 3] using the theory of the hydrogen atom in a magnetic field [1, 4] and the theory of the hydrogen molecule formation as a starting platform [5–7]. The authors of [1, 3] supposed that the holes are sufficiently heavy so as to neglect their Landau quantization states.

At small magnetic fields, the ground state of the hydrogen-like biexciton with the molecular axis directed along the magnetic field corresponds to singlet term ${}^{1}\Sigma_{g}$. With increasing magnetic field, at intermediary values, triplet state ${}^{3}\Sigma_{u}$ becomes lower due to the interaction of the electron spins with the magnetic field. With a further increase in the magnetic field strength, triplet state ${}^{3}\Pi_{u}$ comes into play and becomes the lowest tightly bound hydrogen molecular state [2, 3]. The authors of [1, 6, 7] suggested the possible existence of new bound complexes and chains at fairly high magnetic fields and electron–hole (e–h) densities. The binding energy per one e–h pair in these states may be higher than that in the exciton or excitonic molecule states.

According to [8, 9], the Coulomb exchange electron-hole interaction leads to the formation of para and ortho exciton states, which affect the binding energy of the biexciton. In the case of 2D magnetoexcitons, these aspects have not yet been studied because the main task in the creation of the bimagnetoexciton is the character of the spinless magnetoexciton–magnetoexciton interaction. Already in [10, 11, 12] it was established that the magnetoexcitons composed of electrons and holes lying on the lowest Landau levels (LLLs), being bound in the states with in-plane wave vectors $\vec{k}_{\|}=0$, form an ideal, noninteracting gas. This effect takes place because the 2D electron and the hole moving with resultant wave vector $\vec{k}_{\|}=0$ on the surface of a layer in a perpendicular magnetic field in the Landau gauge description are subjected to the Landau quantizations with the same gyration points. They have the quantum orbits with the same radii, which do not depend on the electron and hole masses and depend only on the magnetic length. These e–h pairs being bound by the direct Coulomb interaction and forming the bound states with wave vector $\vec{k}_{\|}=0$ look as completely neutral compound particles. These magnetoexcitons with $\vec{k}_{\|}=0$ cannot form bound states. Only two magnetoexcitons with opposite nonzero wave vectors $\vec{k}$ and $-\vec{k}$ can form bound states with the resultant wave vector equal to zero. This statement will be investigated in the present paper.

It is useful to remember that the interaction between two magnetoexcitons with wave vectors $\vec{k}_{\|}=0$ can appear, as was shown in [13], taking into account the influence of the excited Landau levels (ELLs) as well as the Rashba spin–orbit coupling (RSOC) generated by a perpendicular external electric field, parallel to the magnetic field. However, all interactions and bound states investigated below have nothing to do with these supplementary influences and are based exclusively on the direct Coulomb interactions of electrons and holes with arbitrary masses situated on their LLLs. The paper is organized as follows. In the second section, the Hamiltonian of the 2D electrons with spins and the heavy holes with effective spins lying on their LLLs and interacting through their Coulomb interactions is introduced. The necessary wave functions are proposed. In section three, the main analytical and numerical calculations are effectuated. Section four is dedicated to the discussions of the obtained results. The conclusions are formulated in section five.

2. The Hamiltonian of the e–h System and the Wave Functions of the Bound States of the Interacting Two-Dimensional Magnetoexcitons

The authors of [10–12, 14, 15] considered spinless 2D electrons and holes paying the main attention to the Landau quantization of their orbital motion on the surface of the layer subjected to the action of an external perpendicular magnetic field. The description of their quantization is made in the Landau gauge, in which the charged particles have a free motion in one in-plane direction described by the plane waves with one-dimensional wave numbers p and q and undergo quantized oscillations around the gyration points in another in-plane direction perpendicular to the previous one.

The quantum numbers of the Landau quantized levels for electrons and holes are n_e and n_h, which in our case are lowest $n_e = n_h = 0$. The creation and annihilation operators for electrons and for holes are denoted as $a^+_{p,\sigma}, a_{p,\sigma}$ and $b^+_{q,\sigma}, b_{q,\sigma}$, respectively. The operators have a supplementary spin label $\sigma = \pm 1/2$, which describes the spin projections of the conduction electrons and the effective spin of the heavy holes. In these denotations, a simple generalization of the Hamiltonian describing the Coulomb interaction of the 2D electrons and holes with spins situated on their LLLs, neglecting for simplicity the e–h exchange interaction leading to the formation of ortho and para magnetoexcitons, as well as the RSOC, has the form

$$\begin{gathered} H^{LLL}_{Coul} = \frac{1}{2}\sum_{\vec{Q}} W(\vec{Q})[\hat{\rho}(\vec{Q})\hat{\rho}(-\vec{Q}) - \hat{N}_e - \hat{N}_h], \\ \hat{\rho}(\vec{Q}) = \hat{\rho}_e(\vec{Q}) - \hat{\rho}_h(\vec{Q}); W(\vec{Q}) = \frac{2\pi e^2}{\varepsilon_0 S|\vec{Q}|} e^{-\frac{Q^2 l_0^2}{2}}, \\ \hat{\rho}_e(\vec{Q}) = \sum_{t,\sigma} e^{iQ_y t l_0^2} a^+_{t+\frac{Q_x}{2},\sigma} a_{t-\frac{Q_x}{2},\sigma}; \hat{N}_e = \hat{\rho}_e(0), \\ \hat{\rho}_h(\vec{Q}) = \sum_{t,\sigma} e^{-iQ_y t l_0^2} b^+_{t+\frac{Q_x}{2},\sigma} b_{t-\frac{Q_x}{2},\sigma}; \hat{N}_h = \hat{\rho}_h(0), \end{gathered} \tag{1}$$

where ε_0 is the dielectric constant, S is the layer surface area, l_0 is the magnetic length, $\hat{\rho}_e(\vec{Q})$ and $\hat{\rho}_h(\vec{Q})$ are the electron and hole plasmon operators, respectively.

Hamiltonian (1) can be transcribed in the way

$$\begin{gathered} H^{LLL}_{Coul} = H^{LLL}_{e-e} + H^{LLL}_{h-h} + H^{LLL}_{e-h}, \\ H^{LLL}_{e-e} = \frac{1}{2}\sum_{\vec{Q}}\sum_{p,q}\sum_{\sigma_1,\sigma_2} W(\vec{Q}) e^{-iQ_xQ_y l_0^2} e^{iQ_y(p-q)l_0^2} a^+_{p,\sigma_1} a^+_{q,\sigma_2} a_{q+Q_x,\sigma_2} a_{p-Q_x,\sigma_1}, \\ H^{LLL}_{h-h} = \frac{1}{2}\sum_{\vec{Q}}\sum_{p,q}\sum_{\sigma_1,\sigma_2} W(\vec{Q}) e^{iQ_xQ_y l_0^2} e^{-iQ_y(p-q)l_0^2} b^+_{p,\sigma_1} b^+_{q,\sigma_2} b_{q+Q_x,\sigma_2} b_{p-Q_x,\sigma_1}, \\ H^{LLL}_{e-h} = -\sum_{\vec{Q}}\sum_{p,q}\sum_{\sigma_1,\sigma_2} W(\vec{Q}) e^{iQ_y(p+q)l_0^2} a^+_{p,\sigma_1} b^+_{q,\sigma_2} b_{q+Q_x,\sigma_2} a_{p-Q_x,\sigma_1}. \end{gathered} \tag{2}$$

The interaction coefficients depend only on the difference ($p-q$) in the case of the electron–electron ($e-e$) and hole–hole ($h-h$) interactions, and on the sum ($p+q$) in the case of the $e-h$ interactions. The magnetoexciton creation operator introduced in [12] and later used in [13–15] with spin labels looks as follows:

$$\hat{\psi}^+_{ex}(\vec{k}, \Sigma_e, \Sigma_h) = \frac{1}{\sqrt{N}}\sum_t e^{ik_y t l_0^2} a^+_{t+\frac{k_x}{2},\Sigma_e} b^+_{-t+\frac{k_x}{2},\Sigma_h}, \quad N = \frac{S}{2\pi l_0^2}; \quad l_0 = \sqrt{\frac{\hbar c}{eB}}. \tag{3}$$

Here, $\vec{k}(k_x, k_y)$ is the vector of the center of mass in-plane motion, t is the unidimensional vector of the relative $e-h$ motion with the function of relative motion $e^{ik_y t l_0^2}$ in the momentum representation, which leads to the $\delta(y-k_y l_0^2)$ function of the relative motion in the real space representation. N is the degree of degeneracy of the Landau quantization levels. It is an extensive value proportional to S. B is the magnetic field strength.

The wave function of the magnetoexciton looks as follows:

$$\left|\psi_{ex}\left(\vec{k},\Sigma_e,\Sigma_h\right)\right\rangle = \hat{\psi}_{ex}^{+}\left(\vec{k},\Sigma_e,\Sigma_h\right)\left|0\right\rangle;\ a_{t,\sigma}\left|0\right\rangle; b_{t,\sigma}\left|0\right\rangle = 0, \tag{4}$$

where $\left|0\right\rangle$ is the ground state of the system. The 2D magnetoexciton with wave vector $\vec{k}\neq 0$ has the form of an electric dipole with arm $d = kl_0^2$ oriented perpendicular to wave vector $\vec{k}$. As was established in [10–12] and was mentioned above, two magnetoexcitons with wave vectors $\vec{k}=0$ have no dipole moments, are similar with the neutral compound particles and do not interact with each other through the Coulomb forces. In contrast to them, two magnetoexcitons with nonzero wave vectors $\vec{k}_1$ and $\vec{k}_2$ do interact opening the possibility of forming bimagnetoexcitons. The wave function of two magnetoexcitons with quantum numbers $\left|\vec{k},\Sigma_{e1},\Sigma_{h1}\right\rangle$ and $\left|-\vec{k},\Sigma_{e2},\Sigma_{h2}\right\rangle$ has the construction

$$\left|\psi_{ex,ex}\left(\vec{k},\Sigma_{e1},\Sigma_{h1};-\vec{k},\Sigma_{e2},\Sigma_{h2}\right)\right\rangle = \frac{1}{N}\sum_{t,s} e^{ik_y(t-s)l_0^2} a^{+}_{t+\frac{k_x}{2},\Sigma_{e1}} a^{+}_{s-\frac{k_x}{2},\Sigma_{e2}} b^{+}_{-s-\frac{k_x}{2},\Sigma_{h2}} b^{+}_{-t+\frac{k_x}{2},\Sigma_{h1}}\left|0\right\rangle. \tag{5}$$

The wave function of the bimagnetoexciton with wave vector $\vec{k}=0$ as a bound state of two magnetoexcitons with wave vectors $\vec{k}$ and $-\vec{k}$ and spin quantum numbers $\Sigma_{e1},\Sigma_{h1},\Sigma_{e2},\Sigma_{h2}$ can be constructed as a superposition of the wave functions (5) introducing wave function $\varphi_n(\vec{k})$ of relative motion, which can also play the role of the variational function determining the minimal energy of the bimagnetoexciton as well as the density $\left|\varphi_n(\vec{k})\right|^2$ of the magnetoexcitons taking part in the formation of the bound state. According to [8, 9], the spin configurations of the bound states depend essentially on the ratio between the ortho–para exciton splitting and the binding energy of the biexciton. In the presence of a strong magnetic field and magnetoexciton formation, these values are unknown; the present paper trends to determine one of them. We will construct symmetric and antisymmetric superpositions of two electron spin states and separately of two hole effective spin states in the form

$$\frac{1}{\sqrt{2}}\sum_{\Sigma_e=\pm\frac{1}{2}}(\eta_e)^{\Sigma_e+\frac{1}{2}} a^{+}_{p,\Sigma_e} a^{+}_{q,-\Sigma_e};\quad \frac{1}{\sqrt{2}}\sum_{\Sigma_h=\pm\frac{1}{2}}(\eta_h)^{\Sigma_h+\frac{1}{2}} b^{+}_{p,\Sigma_h} b^{+}_{q,-\Sigma_h};\quad \eta_e=\pm 1;\quad \eta_h=\pm 1; \tag{6}$$

because the e–h exchange interaction is neglected in the Hamiltonians (1) and (2). Below, we will suppose that both pairs of spins $(\Sigma_{e1},\Sigma_{e2})$ and $(\Sigma_{h1},\Sigma_{h2})$ are simultaneously in the states with the same $\eta_e=\eta_h=\eta=\pm 1$. The wave functions of the bimagnetoexcitons under these conditions look as follows:

$$\left|\psi_{bimex}\left(0,\eta,\varphi_n\right)\right\rangle = \frac{1}{2N^{3/2}}\sum_{\Sigma_e,\Sigma_h}(\eta)^{\Sigma_e+\Sigma_h+1}\sum_{\vec{k}}\varphi_n(\vec{k})\sum_{s,t} e^{ik_y(t-s)l_0^2} a^{+}_{t+\frac{k_x}{2},\Sigma_e} a^{+}_{s-\frac{k_x}{2},-\Sigma_e} b^{+}_{-s-\frac{k_x}{2},-\Sigma_h} b^{+}_{-t+\frac{k_x}{2},\Sigma_h}\left|0\right\rangle. \tag{7}$$

The chosen variational wave functions of the relative motion in the momentum and in the real space representations $\varphi_n(\vec{k})$ and $\psi_n(\vec{r})$, their normalization conditions and the main parameters are the followings:

$$\varphi_0(x) = (4\alpha)^{1/2} e^{-\alpha x^2};\ \varphi_2(x) = (8\alpha^3)^{1/2} x^2 e^{-\alpha x^2};\ x = kl_0,\ \frac{1}{N}\sum_{\vec{k}} \left|\varphi_n(\vec{k})\right|^2 = \int_0^\infty xdx \left|\varphi_n(x)\right|^2 = 1,$$

$$\psi_n(\vec{r}) = \int \varphi_n(\vec{k}) e^{i\vec{k}\vec{r}} d^2\vec{k} = \int_0^\infty xdx\varphi_n(x) J_0(x \cdot r/l_0),\ \psi_0(r) \sim e^{-\frac{r^2}{4\alpha l_0^2}};\ \psi_2(r) \sim (1 - \frac{r^2}{4\alpha l_0^2}) e^{-\frac{r^2}{4\alpha l_0^2}}. \quad (8)$$

Here, $J_0(z)$ is the Bessel function of the zeroth order. The selected trial wave functions depend only on modulus $k = \left|\vec{k}\right|$. $\varphi_0(x)$ has the maximum at the point of $x = 0$, the middle value of $\overline{x^2} = 1/(2\alpha)$, the radius of quantum state $\psi_0(\vec{r})$ equals to $a = 2\sqrt{\alpha} l_0$. Function $\varphi_2(\vec{k})$ has a maximum on the 2D ring with radius $k_r = 1/(\sqrt{\alpha} l_0)$. In the real space, function $\psi_2(r)$ has a maximum at the point of $r_0 = 0$, the positive values up till the point of $r_1 = a_b$, where it changes the sign and achieves a minimum at the point of $r_2 = \sqrt{8\alpha} l_0$. The absolute value in the minimum is much smaller than in the maximum. In fact, function $\psi_2(r)$ has the same radius a_b; however, function $\varphi_2(x)$ has some preferential properties, when the normalization integral of the bimagnetoexciton wave function (7) is calculated. These calculations give rise to the overlapping integrals $L_n(\alpha)$ as follows:

$$\left\langle \phi_{bimex}(0,\eta,\varphi_n) \middle| \phi_{bimex}(0,\eta,\varphi_n) \right\rangle = 2(1 - \eta L_n(\alpha)),$$

$$L_n(\alpha) = \int_0^\infty xdx \int_0^\infty ydy \varphi_n^*(x)\varphi_n(y) J_0(xy);\ L_0(\alpha) = \frac{1}{\alpha + \frac{1}{4\alpha}};\ L_2(\alpha) = \frac{2\alpha^2 - \frac{1}{\alpha}}{\left(\alpha + \frac{1}{4\alpha}\right)^3}. \quad (9)$$

The normalization integrals $(1 - \eta L_n(\alpha))$ are shown in Fig. 1. It is evident that the factor $(1 - L_0(\alpha))$ has the value zero at the point of $\alpha = 1/2$, which leads to a singularity in the inverse function. The inverse normalization integral $(1 - L_2(\alpha))^{-1}$ is regular at any values of α.

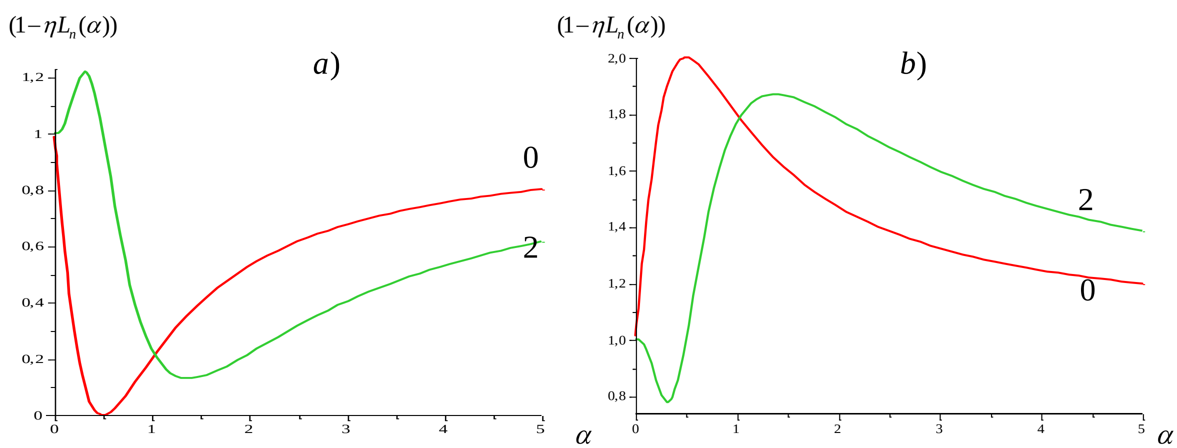

Fig. 1. Normalization integrals as a function of parameter α: (a) the case of $\eta = 1$, (b) the case of $\eta = -1$.

3. Binding Energies of the Bound States of Two Interacting Magnetoexcitons

The expectation values of Hamiltonian (2) averaged with wave function (7) characterized by wave vector $\vec{k}=0$, two values of $\eta=\pm 1$, and by trial wave functions $\varphi_n(\vec{k})$ equal to

$$E_{bimex}\left(0,\eta,\varphi_n\right)=\frac{\left\langle \phi_{bimex}\left(0,\eta,\varphi_n\right)\right|H_{Coul}^{LLL}\left|\phi_{bimex}\left(0,\eta,\varphi_n\right)\right\rangle}{\left\langle \phi_{bimex}\left(0,\eta,\varphi_n\right)\middle|\phi_{bimex}\left(0,\eta,\varphi_n\right)\right\rangle}. \tag{10}$$

The average values of the partial Hamiltonians H_{e-e}^{LLL} and H_{h-h}^{LLL} calculated with the functions (7) are the same and equal to

$$\begin{aligned}&\left\langle \psi_{bimex}\left(0,\eta,\varphi_n\right)\right|H_{e-e}^{LLL}\left|\psi_{bimex}\left(0,\eta,\varphi_n\right)\right\rangle=\left\langle \psi_{bimex}\left(0,\eta,\varphi_n\right)\right|H_{h-h}^{LLL}\left|\psi_{bimex}\left(0,\eta,\varphi_n\right)\right\rangle=\\&=(2/N)\sum_{\vec{Q}}\sum_{\vec{\varkappa}}W\left(\vec{Q}\right)\varphi_n^*\left(\vec{\varkappa}\right)\varphi_n\left(\vec{Q}-\vec{\varkappa}\right)e^{i\left(\varkappa_xQ_y-\varkappa_yQ_x\right)l_0^2}-\\&-(2\eta/N^2)\sum_{\vec{Q}}\sum_{\vec{\varkappa}}\sum_{\vec{k}}W\left(\vec{Q}\right)\varphi_n^*\left(\vec{\varkappa}\right)\varphi_n\left(\vec{k}\right)\exp\left\{il_0^2\left[\left(Q_xk_y-Q_yk_x\right)+\left(\varkappa_xQ_y-\varkappa_yQ_x\right)+\left(k_x\varkappa_y-k_y\varkappa_x\right)\right]\right\}.\end{aligned} \tag{11}$$

In the polar coordinates representation, we can write

$$\begin{aligned}&\vec{k}=\left(k_x,k_y\right)=k\left(\cos\varphi,\sin\varphi\right);\ \vec{k}l_0=\vec{z},\ kl_0=z;\\&\vec{Q}=\left(Q_x,Q_y\right)=Q\left(\cos\theta,\sin\theta\right);\ \vec{Q}l_0=\vec{y};\ Ql_0=y;\\&\vec{\varkappa}=\left(\varkappa_x,\varkappa_y\right)=\varkappa\left(\cos\psi,\sin\psi\right);\ \vec{\varkappa}l_0=\vec{x};\ \varkappa l_0=x;\\&\left(Q_xk_y-Q_yk_x\right)l_0^2=Qkl_0^2\left(\sin\varphi\cos\theta-\cos\varphi\sin\theta\right)=yz\sin\left(\varphi-\theta\right)=z_1\sin t_1;\\&\left(\varkappa_xQ_y-\varkappa_yQ_x\right)l_0^2=Q\varkappa l_0^2\left(\sin\theta\cos\psi-\cos\theta\sin\psi\right)=xy\sin\left(\theta-\psi\right)=z_2\sin t_2;\\&\left(k_x\varkappa_y-k_y\varkappa_x\right)l_0^2=k\varkappa l_0^2\left(\sin\psi\cos\varphi-\cos\psi\sin\varphi\right)=xz\sin\left(\psi-\varphi\right)=z_3\sin t_3.\end{aligned} \tag{12}$$

Here, we have introduced the denotations

$$t_1=\varphi-\theta;\ t_2=\theta-\psi;\ t_3=\psi-\varphi;\ z_1=yz;\ z_2=xy;\ z_3=xz. \tag{13}$$

In these denotations, expression (11) looks as follows:

$$\begin{aligned}&\left\langle \psi_{bimex}\left(0,\eta,\varphi_n\right)\right|H_{j-j}^{LLL}\left|\psi_{bimex}\left(0,\eta,\varphi_n\right)\right\rangle=\frac{2}{N}\sum_{\vec{Q}}\sum_{\vec{\varkappa}}W\left(\vec{Q}\right)\varphi_n^*\left(\vec{\varkappa}\right)\varphi_n\left(\vec{Q}-\vec{\varkappa}\right)e^{iz_2\sin t_2}-\\&-\frac{2\eta}{N^2}\sum_{\vec{Q}}\sum_{\vec{\varkappa}}W\left(\vec{Q}\right)\varphi_n^*\left(\vec{\varkappa}\right)\varphi_n\left(\vec{k}\right)\exp\left(iz_1\sin t_1+iz_2\sin t_2+iz_3\sin t_3\right),\ j=e,h.\end{aligned} \tag{14}$$

In the same denotations, the average e–h Hamiltonian looks as follows:

$$\begin{aligned}&\left\langle \psi_{bimex}\left(0,\eta,\varphi_n\right)\right|H_{e-h}^{LLL}\left|\psi_{bimex}\left(0,\eta,\varphi_n\right)\right\rangle=-(4/N)\sum_{\vec{Q}}\sum_{\vec{\varkappa}}W(\vec{Q})\varphi_n^*(\vec{\varkappa})\varphi_n(\vec{Q}-\vec{\varkappa})-\\&-(4/N)\sum_{\vec{Q}}\sum_{\vec{\varkappa}}W(\vec{Q})\left|\varphi_n\left(\vec{\varkappa}\right)\right|^2\cos\left(z_2\sin t_2\right)+\\&+(4\eta/N^2)\sum_{\vec{Q}}\sum_{\vec{k}}\sum_{\vec{\varkappa}}W(\vec{Q})\varphi_n^*\left(\vec{\varkappa}\right)\varphi_n\left(\vec{k}\right)\left[\cos\left(z_2\sin t_2\right)\cos\left(z_3\sin t_3\right)+\cos\left(z_1\sin t_1\right)\cos\left(z_3\sin t_3\right)\right].\end{aligned} \tag{15}$$

The average value of the full Coulomb interaction Hamiltonian (2) can be expressed as follows:

$$\begin{aligned}&\left\langle \psi_{bimex}\left(0,\eta,\varphi_n\right)\right|H_{Coul}^{LLL}\left|\psi_{bimex}\left(0,\eta,\varphi_n\right)\right\rangle=(4/N)\sum_{\vec{Q}}\sum_{\vec{\varkappa}}W(\vec{Q})\varphi_n^*(\vec{\varkappa})\times\\&\times\varphi_n(\vec{Q}-\vec{\varkappa})(e^{iz_2\sin t_2}-1)-(4/N)\sum_{\vec{Q}}\sum_{\vec{\varkappa}}W(\vec{Q})\left|\varphi_n\left(\vec{\varkappa}\right)\right|^2\times\end{aligned}$$

$$\times\cos\left(z_2\sin t_2\right)+(4\eta/N)\sum_{\vec{Q}}\sum_{\vec{\varkappa}}\sum_{\vec{k}}W\left(\vec{Q}\right)\varphi_n^*\left(\vec{\varkappa}\right)\varphi_n\left(\vec{k}\right)\times$$
$$\times\left[\cos\left(z_2\sin t_2\right)\cos\left(z_3\sin t_3\right)+\cos\left(z_1\sin t_1\right)\cos\left(z_3\sin t_3\right)-\right. \tag{16}$$
$$\left.-\exp\left(iz_1\sin t_1+iz_2\sin t_2+iz_3\sin t_3\right)\right].$$

To calculate these expressions, the relations below are required [16–17]:

$$e^{iz\sin t}=J_0\left(z\right)+2\sum_{k=1}^{\infty}\left[J_{2k}\left(z\right)\cos\left(2kt\right)+iJ_{2k-1}\left(z\right)\sin\left(2k-1\right)t\right];$$
$$\cos\left(z\sin t\right)=J_0\left(z\right)+2\sum_{k=1}^{\infty}J_{2k}\left(z\right)\cos(2kt);\; e^{z\cos t}=I_0\left(z\right)+2\sum_{k=1}^{\infty}I_k\left(z\right)\cos kt, \tag{17}$$

where $J_{\upsilon}(z)$ and $I_{\upsilon}(z)$ are the Bessel functions [16–17]. Taking into account that functions $W(\vec{Q})$, $\varphi_n\left(\vec{\varkappa}\right)$ and $\varphi_n(\vec{k})$ depend only on moduli $|\vec{Q}|$, $|\vec{\varkappa}|$ and $|\vec{k}|$, respectively, the integrations on angles φ, θ, and ψ lead to the results

$$\frac{1}{2\pi}\int_0^{2\pi}d\varphi\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{1}{2\pi}\int_0^{2\pi}d\psi e^{iz_1\sin t_1}e^{iz_2\sin t_2}e^{iz_3\sin t_3}=J_0\left(z_1\right)J_0\left(z_2\right)J_0\left(z_3\right)+2\sum_{k=1}^{\infty}J_{2k}\left(z_1\right)J_{2k}\left(z_2\right)J_{2k}\left(z_3\right);$$
$$\frac{1}{2\pi}\int_0^{2\pi}d\varphi\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{1}{2\pi}\int_0^{2\pi}d\psi\cos\left(z_i\sin t_i\right)\cos\left(z_3\sin t_3\right)=J_0\left(z_i\right)J_0\left(z_3\right),\; i=1,2;$$
$$\frac{1}{2\pi}\int_0^{2\pi}d\varphi\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{1}{2\pi}\int_0^{2\pi}d\psi\cos\left(z_i\sin t_i\right)=J_0\left(z_i\right),\; i=1,2,3;$$
$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{1}{2\pi}\int_0^{2\pi}d\psi e^{2\alpha xy\cos\left(\theta-\psi\right)}e^{ixy\sin\left(\theta-\psi\right)}=J_0\left(xy\right)I_0\left(2\alpha xy\right)+2\sum_{k=1}^{\infty}J_{2k}\left(xy\right)I_{2k}\left(2\alpha xy\right);$$
$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{1}{2\pi}\int_0^{2\pi}d\psi e^{2\alpha xy\cos\left(\theta-\psi\right)}=I_0\left(2\alpha xy\right). \tag{18}$$

The angle integration excluding trial function $\varphi_n\left(\left|\vec{x}-\vec{y}\right|\right)$ leads to the expression

$$\left\langle\psi_{bimex}\left(0,\eta,\varphi_n\right)\right|H_{coul}^{LLL}\left|\psi_{bimex}\left(0,\eta,\varphi_n\right)\right\rangle=-4(e^2/(\varepsilon_0 l_0))\int_0^{\infty}dye^{-y^2/2}\times$$
$$\times\int_0^{\infty}xdx\left|\varphi_n\left(x\right)\right|^2J_0\left(xy\right)+4(e^2/(\varepsilon_0 l_0))\int_0^{\infty}dye^{-y^2/2}\int_0^{\infty}xdx\varphi_n^*\left(x\right)\frac{1}{2\pi}\int_0^{2\pi}d\theta\times$$
$$\times\frac{1}{2\pi}\int_0^{2\pi}d\psi\varphi_n\left(\left|\vec{x}-\vec{y}\right|\right)\left(e^{ix\cdot y\sin\left(\theta-\psi\right)}-1\right)+4\eta(e^2/(\varepsilon_0 l_0))\int_0^{\infty}dye^{-y^2/2}\times$$
$$\times\int_0^{\infty}xdx\varphi_n^*\left(x\right)\int_0^{\infty}zdz\varphi_n\left(z\right)\left(J_0\left(x\cdot y\right)J_0\left(x\cdot z\right)+J_0\left(x\cdot z\right)J_0\left(y\cdot z\right)-\right. \tag{19}$$
$$\left.-J_0\left(x\cdot y\right)J_0\left(x\cdot z\right)J_0\left(y\cdot z\right)-2\sum_{k=1}^{\infty}J_{2k}\left(x\cdot y\right)J_{2k}\left(x\cdot z\right)J_{2k}\left(y\cdot z\right)\right),$$

where $\left|\vec{x}-\vec{y}\right|=\sqrt{x^2+y^2-2xy\cos\left(\theta-\psi\right)}$.

Trial wave function $\varphi_2\left(x\right)$ can be represented in the form

$$\varphi_2\left(\left|\vec{x}-\vec{y}\right|\right)=\left(8\alpha^3\right)^{1/2}\left|\vec{x}-\vec{y}\right|^2 e^{-\alpha\left|\vec{x}-\vec{y}\right|^2}=\left(8\alpha^3\right)^{1/2}e^{-\alpha\left(x^2+y^2\right)}\times$$
$$\times\left[x^2+y^2-2xy\frac{\partial}{\partial\left(2\alpha xy\right)}\right]\left[I_0\left(2\alpha xy\right)+2\sum_{k=1}^{\infty}I_{2k}\left(2\alpha xy\right)\cos 2k\left(\theta-\varphi\right)\right]. \tag{20}$$

Using integrals (18), one can obtain the expression

$$\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{1}{2\pi}\int_0^{2\pi}d\psi\varphi_2\left(\left|\vec{x}-\vec{y}\right|\right)\left(e^{ixy\sin(\theta-\psi)}-1\right)=\left(8\alpha^3\right)^{1/2}e^{-\alpha\left(x^2+y^2\right)}\times$$
$$\times\left\{\left(x^2+y^2\right)\left[\left(J_0\left(xy\right)-1\right)I_0\left(2\alpha xy\right)+\sum_{k=1}^{\infty}J_{2k}\left(xy\right)I_{2k}\left(2\alpha xy\right)\right]-\right.$$
$$-2xy\left[\left(J_0\left(xy\right)-1\right)I_1\left(2\alpha xy\right)+\sum_{k=1}^{\infty}J_{2k}\left(xy\right)I_{2k+1}\left(2\alpha xy\right)\right]- \tag{21}$$
$$\left.-\frac{2}{\alpha}\sum_{k=1}^{\infty}k\cdot J_{2k}\left(xy\right)I_{2k}\left(2\alpha xy\right)\right\}.$$

Here, the derivatives of Bessel functions $I_n\left(z\right)$ and $J_n(z)$ of the integer order were used [16, 18]:

$$\frac{dI_n\left(z\right)}{dz}=\frac{n}{z}I_n\left(z\right)+I_{n+1}\left(z\right);\ \frac{dJ_n\left(z\right)}{dz}=\frac{n}{z}J_n\left(z\right)-J_{n+1}\left(z\right). \tag{22}$$

Taking into account the integral (21), the second term in the right-hand side of the average value (19) can be transcribed in the following way

$$\varepsilon_2=4\left(\frac{e^2}{\varepsilon_0 l_0}\right)\int_0^{\infty}dye^{-\frac{y^2}{2}}\int_0^{\infty}xdx\varphi_2^*\left(x\right)\frac{1}{2\pi}\int_0^{2\pi}d\theta\frac{1}{2\pi}\int_0^{2\pi}d\psi\times$$
$$\times\varphi_2\left(\left|\vec{x}-\vec{y}\right|\right)\left(e^{ixy\sin(\theta-\psi)}-1\right)=4\left(\frac{e^2}{\varepsilon_0 l_0}\right)\left(8\alpha^3\right)\times$$
$$\times\left\{\int_0^{\infty}dye^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^{\infty}dxx^5e^{-2\alpha x^2}J_0\left(x\cdot y\right)I_0\left(2\alpha xy\right)+\int_0^{\infty}dyy^2e^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^{\infty}dxx^3e^{-2\alpha x^2}J_0\left(xy\right)I_0\left(2\alpha xy\right)-\right.$$
$$-\int_0^{\infty}dye^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^{\infty}dxx^5e^{-2\alpha x^2}I_0\left(2\alpha xy\right)-2\int_0^{\infty}dy\cdot ye^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^{\infty}dxx^4e^{-2\alpha x^2}J_0\left(xy\right)I_1\left(2\alpha xy\right)-$$
$$-\int_0^{\infty}dyy^2e^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^{\infty}dxx^3e^{-2\alpha x^2}I_0\left(2\alpha xy\right)+2\int_0^{\infty}dy\cdot ye^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^{\infty}dxx^4e^{-2\alpha x^2}I_1\left(2\alpha xy\right)+$$
$$+2\int_0^{\infty}dye^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^{\infty}dx\cdot x^5e^{-2\alpha x^2}\sum_{k=1}^{\infty}J_{2k}\left(xy\right)I_{2k}\left(2\alpha xy\right)-$$
$$-4\int_0^{\infty}dy\cdot ye^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^{\infty}dxx^4e^{-2\alpha x^2}\sum_{k=1}^{\infty}J_{2k}\left(xy\right)I_{2k+1}\left(2\alpha xy\right)+$$
$$+2\int_0^{\infty}dyy^2e^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^{\infty}dxx^3e^{-2\alpha x^2}\sum_{k=1}^{\infty}J_{2k}\left(xy\right)I_{2k}\left(2\alpha xy\right)-$$

$$-\frac{4}{\alpha}\int_0^\infty dye^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^\infty dxx^3e^{-2\alpha x^2}\sum_{k=1}^\infty kJ_{2k}\left(xy\right)I_{2k}\left(2\alpha xy\right)\Bigg\}. \tag{23}$$

The third contribution to the average value (19) looks as follows:

$$\begin{aligned}\varepsilon_3 &= 4\eta\left(\frac{e^2}{\varepsilon_0 l_0}\right)\int_0^\infty dye^{-\frac{y^2}{2}}\int_0^\infty xdx\varphi_2^*\left(x\right)\int_0^\infty zdz\varphi_2\left(z\right)\left(J_0\left(x\cdot y\right)J_0\left(x\cdot z\right)+J_0\left(x\cdot z\right)J_0\left(y\cdot z\right)-\right.\\ &\left.-J_0\left(x\cdot y\right)J_0\left(x\cdot z\right)J_0\left(y\cdot z\right)-2\sum_{k=1}^\infty J_{2k}\left(x\cdot y\right)J_{2k}\left(x\cdot z\right)J_{2k}\left(y\cdot z\right)\right)=\\ &=4\eta\left(\frac{e^2}{\varepsilon_0 l_0}\right)\left(8\alpha^3\right)\int_0^\infty dye^{-\frac{y^2}{2}}\int_0^\infty dxx^3e^{-\alpha x^2}\int_0^\infty dzz^3e^{-\alpha z^2}\left[J_0\left(x\cdot y\right)J_0\left(x\cdot z\right)+J_0\left(x\cdot z\right)J_0\left(y\cdot z\right)-\right.\\ &\left.-J_0\left(x\cdot y\right)J_0\left(x\cdot z\right)J_0\left(y\cdot z\right)-2\sum_{k=1}^\infty J_{2k}\left(xy\right)J_{2k}\left(x\cdot z\right)J_{2k}\left(y\cdot z\right)\right].\end{aligned} \tag{24}$$

The first term in the average value (19), as well as the overlapping integrals $L_n(\alpha)$, can be calculated analytically as follows:

$$\varepsilon_1=-4\left(\frac{e^2}{\varepsilon_0 l_0}\right)\int_0^\infty dye^{-\frac{y^2}{2}}\int_0^\infty dxx\left|\varphi_2\left(x\right)\right|^2J_0\left(xy\right)=-4I_l\cdot\frac{\sqrt{4\alpha}}{\sqrt{1+4\alpha}}\left[1-\frac{1}{1+4\alpha}+\frac{3}{8}\frac{1}{\left(1+4\alpha\right)^2}\right];$$

$$I_l=\left(\frac{e^2}{\varepsilon_0 l_0}\right)\sqrt{\frac{\pi}{2}};\quad L_2(\alpha)=\frac{2\alpha^2-\frac{1}{2}}{\left(\alpha+\frac{1}{4\alpha}\right)^3}. \tag{25}$$

Here, I_l is the ionization potential of the 2D magnetoexciton with wave vector $k_{\|}=0$.

$$\frac{E_1(\alpha)}{2I_l}=\frac{\varepsilon_1(\alpha)}{4I_l(1-\eta L_2(\alpha))}=-\frac{\sqrt{4\alpha}\left[1-\frac{1}{1+4\alpha}+\frac{3}{8}\frac{1}{\left(1+4\alpha\right)^2}\right]}{\sqrt{1+4\alpha}\left[1-\frac{\eta(2\alpha^2-1/2)}{\left(\alpha+1/(4\alpha)\right)^3}\right]}. \tag{26}$$

In expression ε_2 (23) there are three integrals containing only one modified Bessel function $I_\nu(2\alpha xy)$ with $\nu=0,1$. They depend on parameter α of variational wave function $\varphi_2\left(x\right)$ in the way

$$\begin{aligned}I_1&=-\int_0^\infty dye^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^\infty dxx^5e^{-2\alpha x^2}I_0\left(2\alpha xy\right)=-\frac{\sqrt{2\pi}}{128\alpha^3}\cdot\frac{\left(8+24\alpha+19\alpha^2\right)}{\left(1+\alpha\right)^{5/2}},\\ I_2&=-\int_0^\infty dyy^2e^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^\infty dxx^3e^{-2\alpha x^2}I_0\left(2\alpha xy\right)=-\frac{\sqrt{2\pi}}{32\alpha^2}\cdot\frac{\left(2+5\alpha\right)}{\left(1+\alpha\right)^{5/2}},\\ I_3&=2\int_0^\infty dy\cdot y\cdot e^{-y^2\left(\frac{1}{2}+\alpha\right)}I_1\left(2\alpha xy\right)=\frac{\sqrt{2\pi}}{32\alpha^2}\cdot\frac{\left(4+7\alpha\right)}{\left(1+\alpha\right)^{5/2}}.\end{aligned} \tag{27}$$

In contrast to the first three integrals I_1, I_2, I_3 containing one modified Bessel function $I_0\left(cx\right)$ or $I_1\left(cx\right)$, the other three integrals contain the products of two Bessel functions of the

types $J_0(bx)I_0(cx)$ and $J_0(bx)I_1(cx)$, one of them being also a modified Bessel function. Integrals I_4, I_5, and I_6 were calculated analytically exactly.
They have the expressions

$$I_4=\int_0^\infty dye^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^\infty dxx^5e^{-2\alpha x^2}J_0(xy)I_0(2\alpha xy)=\frac{1}{32\alpha^4}\left\{3I_0^{1/2}(q,c)+\frac{3}{8\alpha}\left(4\alpha^2-1\right)I_0^{3/2}(q,c)+\right.$$
$$\left.+\frac{\left(4\alpha^2-1\right)}{128\alpha^2}I_0^{5/2}(q,c)-\frac{7}{4}I_1^{3/2}(q,c)-\frac{\left(4\alpha^2-1\right)}{16\alpha}I_1^{5/2}(q,c)+\frac{1}{8}I_2^{5/2}(q,c)\right\},$$
$$I_5=\int_0^\infty dyy^2e^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^\infty dxx^3e^{-2\alpha x^2}J_0(xy)I_0(2\alpha xy)=$$
$$=\frac{1}{16\alpha^2}\left\{I_0^{3/2}(q,c)+\frac{\left(4\alpha^2-1\right)}{8\alpha}I_0^{5/2}(q,c)-\frac{1}{2}I_1^{5/2}(q,c)\right\},$$
$$I_6=-2\int_0^\infty dyye^{-y^2\left(\frac{1}{2}+\alpha\right)}\int_0^\infty dxx^4e^{-2\alpha x^2}J_0(xy)I_1(2\alpha xy)=$$
$$=-\frac{1}{16\alpha^3}\left\{I_0^1(q,c)+\frac{\left(4\alpha^2-1\right)}{16\alpha}I_0^2(q,c)-\frac{1}{4}I_1^2(q,c)-\right.$$
$$\left.-\frac{3}{2}I_1^{3/2}(q,c)-\frac{\left(4\alpha^2-1\right)}{16\alpha}I_1^{5/2}(q,c)+\frac{1}{4}I_2^{5/2}(q,c)\right\}. \tag{28}$$

Here, the following denotations were used [18]:

$$I_0^1(q,c)=\frac{1}{\left(q^2+c^2\right)^{1/2}};\ I_0^2(q,c)=\frac{q}{\left(q^2+c^2\right)^{3/2}};\ I_1^2(q,c)=\frac{c}{\left(q^2+c^2\right)^{3/2}};I_0^{1/2}(q,c)=2\sqrt{\frac{\left(1-2k^2\right)}{\pi q}}K(k),$$
$$I_0^{3/2}(q,c)=\sqrt{\frac{\left(1-2k^2\right)^3}{\pi q^3}}\left[2E(k)-K(k)\right],$$
$$I_0^{5/2}(q,c)=\frac{1}{2}\sqrt{\frac{\left(1-2k^2\right)^5}{\pi q^5}}\left[8\left(1-2k^2\right)E(k)-\left(5-8k^2\right)K(k)\right],$$
$$I_1^{3/2}(q,c)=\frac{1}{k}\sqrt{\frac{\left(1-2k^2\right)^3}{\pi q^3\left(1-k^2\right)}}\left[\left(1-k^2\right)K(k)-\left(1-2k^2\right)E(k)\right],$$
$$I_1^{5/2}(q,c)=\frac{1}{2k}\sqrt{\frac{\left(1-2k^2\right)^5}{\pi q^5\left(1-k^2\right)}}\left[\left(1-k^2\right)\left(1-8k^2\right)K(k)-\left(1-16k^2+16k^4\right)E(k)\right],$$
$$I_2^{5/2}(q,c)=\frac{1}{2k^2\left(1-k^2\right)}\sqrt{\frac{\left(1-2k^2\right)^5}{\pi q^5}}\left[\left(2+5k^2-8k^4\right)\times\right. \tag{29}$$
$$\left.\times\left(1-k^2\right)K(k)-2\left(1-2k^2\right)\left(1+4k^2-4k^4\right)E(k)\right].$$

The complete elliptic integrals of the first and second orders $K(k)$ and $E(k)$ depend on modulus k, which in its turn depends on parameters q and c in the way

$$k=\frac{1}{\sqrt{2}}\left(1-\frac{q}{\sqrt{q^2+c^2}}\right)^{1/2},\ q=\frac{1+4\alpha+4\alpha^2}{8\alpha};\ c=\frac{1}{2}. \tag{30}$$

In the range of small values of the modulus $k \ll 1$ the series expansions of functions $K(k)$ and $E(k)$ can be used:

$$\underset{k\to 0}{K(k)}\approx\frac{\pi}{2}\left(1+\frac{k^2}{4}+\frac{9}{64}k^4\right);\ \underset{k\to 0}{E(k)}\approx\frac{\pi}{2}\left(1-\frac{k^2}{4}-\frac{3}{64}k^4\right). \tag{31}$$

In this limit, the apparent singular expressions in formulas (29) can be transformed in the regular forms as follows:

$$\begin{aligned}&\frac{1}{k}\left[\left(1-k^2\right)K(k)-\left(1-2k^2\right)E(k)\right]\approx\frac{3}{4}\pi k\left(1-\frac{3}{8}k^2\right);\\&\frac{1}{k}\left[\left(1-k^2\right)\left(1-8k^2\right)K(k)-\left(1-16k^2+16k^4\right)E(k)\right]\approx\frac{15}{4}\pi k\left(1-\frac{15}{8}k^2\right);\\&\frac{1}{k^2}\left[\left(2+5k^2-8k^4\right)\left(1-k^2\right)K(k)-2\left(1-2k^2\right)\left(1+4k^2-4k^4\right)E(k)\right]\approx\frac{105}{16}\pi k^2;\ k\to 0.\end{aligned} \tag{32}$$

Using these expressions, we can obtain the simplified formulas in the limit $k\to 0$

$$I_1^{3/2}(q,c)=\frac{3}{4}\pi k\sqrt{\frac{\left(1-2k^2\right)^3}{\pi q^3\left(1-k^2\right)}}\left(1-\frac{3}{8}k^2\right);I_1^{5/2}(q,c)=\frac{15}{8}\pi k\sqrt{\frac{\left(1-2k^2\right)^5}{\pi q^5\left(1-k^2\right)}}\left(1-\frac{15}{8}k^2\right);$$

$$I_2^{5/2}(q,c)=\frac{105}{32}\pi\frac{k^2}{\left(1-k^2\right)}\sqrt{\frac{\left(1-2k^2\right)^5}{\pi q^5}};k\to 0. \tag{33}$$

The required table integrals I_ν^β as a function of parameter α in the case of parameters $q=(1+4\alpha+4\alpha^2)/(8\alpha);c=1/2$ are shown in Fig. 2.

The next three integrals I_7, I_8, and I_9 determine the third contribution ε_3 described by the formulas (24). Some details of their calculations are the following. Integral I_7 equals to

$$\begin{aligned}I_7&=\int_0^\infty dye^{-\frac{y^2}{2}}\int_0^\infty dxx^3e^{-\alpha x^2}\int_0^\infty dzz^3e^{-\alpha z^2}\left(J_0(xy)J_0(xz)+J_0(xz)J_0(yz)\right)=\\&=\frac{4\sqrt{\pi}(4\alpha^2-1)}{\left(4\alpha^2+1\right)^3q^{1/2}}-\frac{2\alpha\sqrt{\pi}(4\alpha^2-3)}{\left(4\alpha^2+1\right)^4q^{3/2}}-\frac{3\alpha^2\sqrt{\pi}}{4\left(4\alpha^2+1\right)^4q^{5/2}}.\end{aligned} \tag{34}$$

It was calculated exactly taking into account that the product of two Bessel functions $J_0(xz)J_0(yz)$ can be transformed into the expression $J_0(xy)J_0(xz)$ by the interchange of the variables $x \rightleftarrows z$. The integral [18] was used:

$$\int_0^\infty dzze^{-pz^2}J_0(xz)=1/(2p)e^{-\frac{x^2}{4p}}. \tag{35}$$

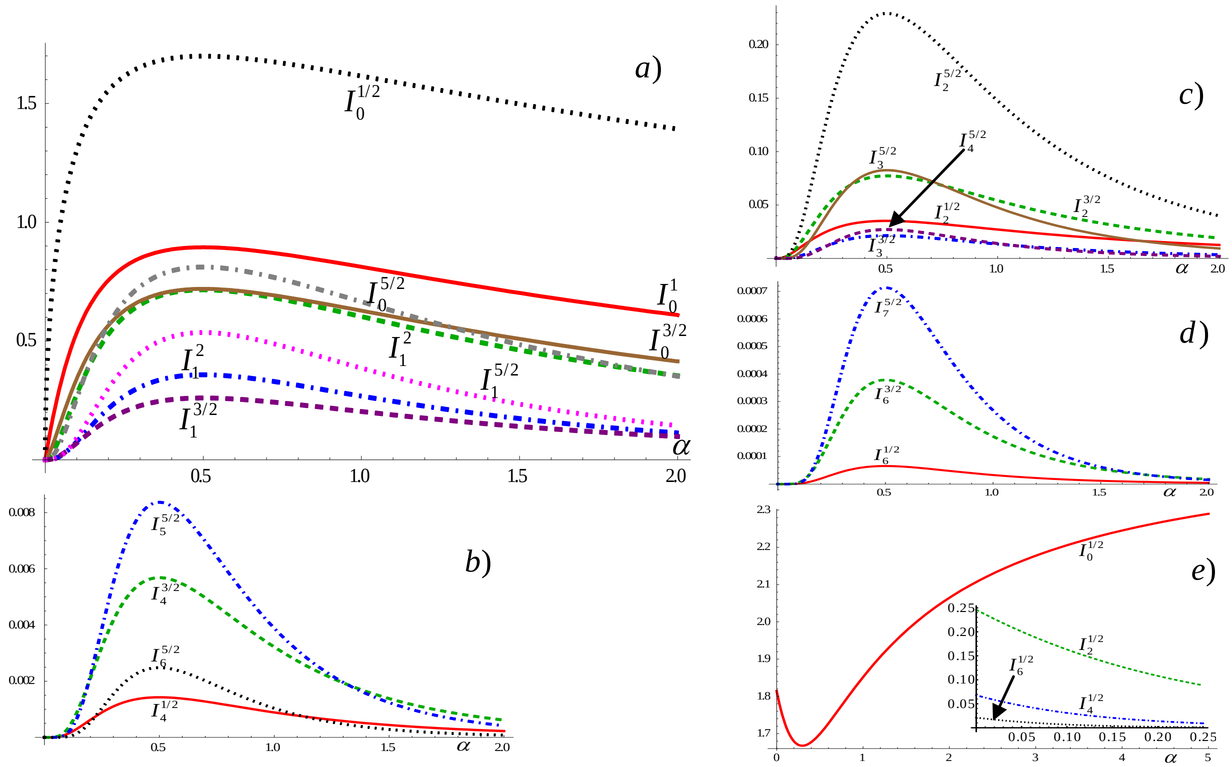

Fig. 2. Table integrals I_ν^β as a function of indices ν and β, and parameters $q=\dfrac{4\alpha^2+4\alpha+1}{8\alpha}, c=\dfrac{1}{2}$ in the cases (a–d) and parameters $q=\dfrac{1+4\alpha+4\alpha^2}{2(1+4\alpha^2)}, c=\dfrac{1}{1+4\alpha^2}$ in the case (e). All of them are plotted as a function of parameter α.

There are two integrals I_8 and I_9 containing three Bessel functions as follows:

$$I_8=-\int_0^\infty dy e^{-\frac{y^2}{2}}\int_0^\infty dx x^3 e^{-\alpha x^2}\int_0^\infty dz z^3 e^{-\alpha z^2} J_0(xy)J_0(xz)J_0(yz), \tag{36}$$

and

$$I_9=-2\sum_{n=1}^\infty\int_0^\infty dy e^{-\frac{y^2}{2}}\int_0^\infty dx x^3 e^{-\alpha x^2}\int_0^\infty dz z^3 e^{-\alpha z^2}\cdot J_{2n}(xy)J_{2n}(xz)J_{2n}(yz). \tag{37}$$

The three steps in the deduction of integrals I_8, I_9 are based on the use of the formulas [17, 18]:

$$\begin{gathered}\int_0^\infty dz z e^{-\alpha z^2} J_0(xz)J_0(yz)=\frac{1}{2\alpha}e^{-\frac{(x^2+y^2)}{4\alpha}} I_0\left(\frac{xy}{2\alpha}\right),\\ \int_0^\infty dx x e^{-px^2} J_\upsilon(bx) I_\upsilon(cx)=\frac{1}{2p}\exp\left(\frac{c^2-b^2}{4p}\right) J_\upsilon\left(\frac{bc}{2p}\right),\\ \int_0^\infty x^{\beta-1}e^{-qx}J_\upsilon(cx)=I_\upsilon^\beta(q,c)\ .\end{gathered} \tag{38}$$

The exact calculations of integrals (36) and (37) leads to cumbersome expressions listed in the appendix. Expressions $I_{2n}^{1/2}\left(q,c\right)$ of the exact formulas (35)–(38), as well as of the appendix, are described in [18]. There are still four double integrals $I_{10}-I_{13}$ in the composition of expression (23). They were calculated analytically exactly below. In all of them the table integral [18]

$$Z_{2n}\left(p,b,c\right)=\int_0^\infty dxxe^{-px^2}J_{2n}\left(bx\right)I_{2n}\left(cx\right)=\frac{1}{2p}e^{\frac{\left(c^2-b^2\right)}{4p}}J_{2n}\left(\frac{bc}{2p}\right) \tag{39}$$

and its derivatives were used as the first step:

$$-\frac{d}{dp}Z_{2n}\left(p,b,c\right)=\frac{1}{2p^2}\exp\left(\frac{c^2-b^2}{4p}\right)\left[\left(2n+1+\frac{c^2-b^2}{4p}\right)J_{2n}\left(\frac{bc}{2p}\right)-\right.$$

$$\left.-\frac{bc}{2p}J_{2n+1}\left(\frac{bc}{2p}\right)\right]=\int_0^\infty dxx^3e^{-px^2}J_{2n}\left(bx\right)I_{2n}\left(cx\right);$$

$$\frac{d^2}{dp^2}Z_{2n}\left(p,b,c\right)=\int_0^\infty dxx^5e^{-px^2}J_{2n}\left(bx\right)I_{2n}\left(c,x\right)=$$

$$=\frac{1}{p^3}e^{\frac{c^2-b^2}{4p}}\left\{J_{2n}\left(\frac{bc}{2p}\right)\left[\left(1+n+\frac{c^2-b^2}{8p}\right)\left(1+2n+\frac{c^2-b^2}{4p}\right)+\frac{c^2-b^2}{8p}\right]-\right.$$

$$\left.-\frac{bc}{4p}\left(4n+5+\frac{c^2-b^2}{2p}\right)J_{2n+1}\left(\frac{bc}{2p}\right)+\frac{b^2c^2}{8p}J_{2n+2}\left(\frac{bc}{2p}\right)\right\};$$

$$\left[-\frac{\partial^2}{\partial p\partial c}+\frac{2n}{c}\frac{\partial}{\partial p}\right]Z_{2n}\left(p,b,c\right)=\int_0^\infty dxx^4e^{-px^2}J_{2n}\left(bx\right)I_{2n+1}\left(cx\right)=$$

$$=\frac{1}{2p^2}e^{\frac{c^2-b^2}{4p}}\left\{\frac{c}{2p}\left[2\left(n+1\right)+\frac{c^2-b^2}{4p}\right]J_{2n}\left(\frac{bc}{2p}\right)-\right.$$

$$\left.-\left[\frac{bc^2}{4p^2}+\frac{b}{2p}\left(2n+3+\frac{c^2-b^2}{4p}\right)\right]J_{2n+1}\left(\frac{bc}{2p}\right)+\frac{b^2c}{4p^2}J_{2n+2}\left(\frac{bc}{2p}\right)\right\}. \tag{40}$$

In all these formulas, it is necessary to substitute parameters $p=2x,\ b=y$, and $c=2\alpha y$. They lead to expressions $(c^2-b^2)/(4p)=y^2(4\alpha^2-1)/(8\alpha)$, $(bc)/(2p)=y^2/2$, $c/(2p)=y/2$; $(bc^2)/(4p^2)=y^3/4$, $(b^2c)/(4p^2)=y^3/(8\alpha)$, $1/(2p^2)=1/(8\alpha^2)$. Formulas (40) permit to calculate the integral

$$\int_0^\infty dxx^4e^{-2\alpha x^2}J_{2n}\left(xy\right)I_{2n+1}\left(2\alpha xy\right)=\frac{1}{8\alpha^2}e^{\frac{\left(4\alpha^2-1\right)}{8\alpha}y^2}\times$$

$$\times\left\{\frac{y}{2}\left[2\left(n+1\right)+\frac{\left(4\alpha^2-1\right)}{8\alpha}y^2\right]J_{2n}\left(\frac{y^2}{2}\right)-\left[\frac{y}{4\alpha}\left(2n+3+\frac{\left(4\alpha^2-1\right)}{8\alpha}y^2\right)\right]\times\right. \tag{41}$$

$$\left.\times J_{2n+1}\left(\frac{y^2}{2}\right)+\frac{y^3}{8\alpha^2}J_{2n+2}\left(\frac{y^2}{2}\right)\right\}.$$

The following integration on variable y using the table integral [18]

$$\int_0^\infty z^{\alpha-1}e^{-qz}J_\upsilon(cz)\,dz = I_\upsilon^\alpha(q,c), \tag{42}$$

gives the possibility of exactly calculating the last four double integrals as follows:

$$I_{10} = 2\sum_{n=1}^\infty \int_0^\infty dy y^2 e^{-y^2\left(\frac{1}{2}+\alpha\right)} \int_0^\infty dx x^3 e^{-2\alpha x^2} J_{2n}(xy) I_{2n}(2\alpha xy) =$$
$$= \frac{1}{8\alpha^2}\sum_{n=1}^\infty \left[(2n+1) I_{2n}^{3/2}(q,c) + \frac{(4\alpha^2-1)}{8\alpha} I_{2n}^{5/2}(q,c) - \frac{1}{2} I_{2n+1}^{5/2}(q,c)\right],$$
$$I_{11} = -\frac{4}{\alpha}\sum_{n=1}^\infty n \int_0^\infty dy e^{-y^2\left(\frac{1}{2}+\alpha\right)} \int_0^\infty dx x^3 e^{-2\alpha x^2} J_{2n}(xy) I_{2n}(2\alpha xy) =$$
$$= -\frac{1}{4\alpha^3}\sum_{n=1}^\infty n\left[(1+2n) I_{2n}^{1/2}(q,c) + \frac{(4\alpha^2-1)}{8\alpha} I_{2n}^{3/2}(q,c) - \frac{1}{2} I_{2n+1}^{3/2}(q,c)\right];$$
$$I_{12} = 2\sum_{n=1}^\infty \int_0^\infty dy e^{-y^2\left(\frac{1}{2}+\alpha\right)} \int_0^\infty dx x^5 e^{-2\alpha x^2} J_{2n}(xy) I_{2n}(2\alpha xy) =$$
$$= \frac{1}{8\alpha^3}\sum_{n=1}^\infty \left[(1+n)(1+2n) I_{2n}^{1/2}(q,c) + (1+n)\frac{(4\alpha^2-1)}{4\alpha} I_{2n}^{3/2}(q,c) + \right.$$
$$\left. + \frac{(4\alpha^2-1)^2}{128\alpha^2} I_{2n}^{5/2}(q,c) - \frac{(4n+5)}{4} I_{2n+1}^{3/2}(q,c) - \frac{(4\alpha^2-1)}{16\alpha} I_{2n+1}^{5/2}(q,c) + \frac{\alpha}{4} I_{2n+2}^{5/2}(q,c)\right];$$
$$I_{13} = -4\sum_{n=1}^\infty \int_0^\infty dy y e^{-y^2\left(\frac{1}{2}+\alpha\right)} \int_0^\infty dx x^4 e^{-2\alpha x^2} J_{2n}(xy) I_{2n+1}(2\alpha xy) =$$
$$= -\frac{1}{4\alpha^2}\sum_{n=1}^\infty \left[(n+1) I_{2n}^{3/2}(q,c) + \frac{(4\alpha^2-1)}{16\alpha} I_{2n}^{5/2}(q,c) - \frac{(2n+3)}{4\alpha} I_{2n+1}^{3/2}(q,c) - \left(\frac{1}{4} + \frac{4\alpha^2-1}{32\alpha^2}\right) I_{2n+1}^{5/2}(q,c) + \right.$$
$$\left. + \frac{1}{8\alpha} I_{2n+2}^{5/2}(q,c)\right]; q = \frac{1+4\alpha+4\alpha^2}{8\alpha}, c = \frac{1}{2}, k = \frac{1}{\sqrt{2}}\left(1 - \frac{q}{\sqrt{q^2+c^2}}\right)^{1/2}. \tag{43}$$

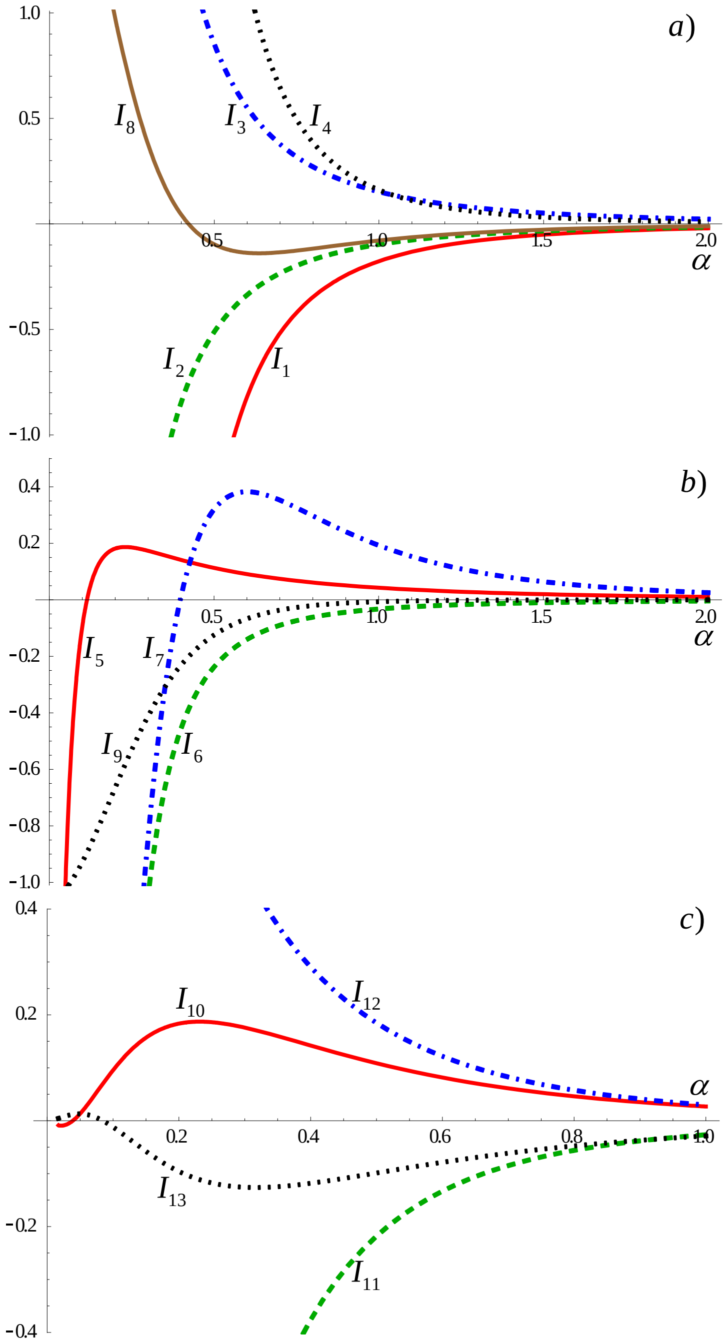

Fig. 3. Integrals $I_1 - I_{13}$ as a function of parameter α of trial wave function $\varphi_2(k)$.

The contributions to the energy spectrum expressed by integrals $I_1 - I_{13}$ are shown in Fig. 3 as the functions of parameter α of trial wave function $\varphi_2(k)$.

4. The Electron Structure of the Bound States

As was shown in the previous papers [10–12], the interaction of two 2D magnetoexcitons with wave vectors $\vec{k} = 0$, which are composed of electrons and holes lying on the LLLs, vanishes because they look as two neutral compound particles. The interactions between them can appear under the influence of the ELLs as well as the RSOC [13]. In the absence of the last interactions, only two magnetoexcitons with wave vectors $\vec{k} \neq 0$ can interact through the Coulomb forces and can take part in the formation of bound states. The 2D magnetoexcitons with $\vec{k} \neq 0$ look as electric dipoles whose the in-plane arms have a length of $d = kl_0^2$, where l_0 is the magnetic length. The arms are oriented perpendicular to the direction of wave vectors $\vec{k}$. The molecule and the bound states can be formed by two magnetoexcitons with antiparallel wave vectors $\vec{k}$ and $-\vec{k}$. They have the structure of two antiparallel dipoles bound together. The possibility of their orientation as a whole in any direction of the layer plane with equal probability was supposed. This possibility is achieved via introducing the trial wave function of relative motion of two magnetoexcitons in the frame of bound state $\varphi_n(\vec{k})$, which depends on modulus k.

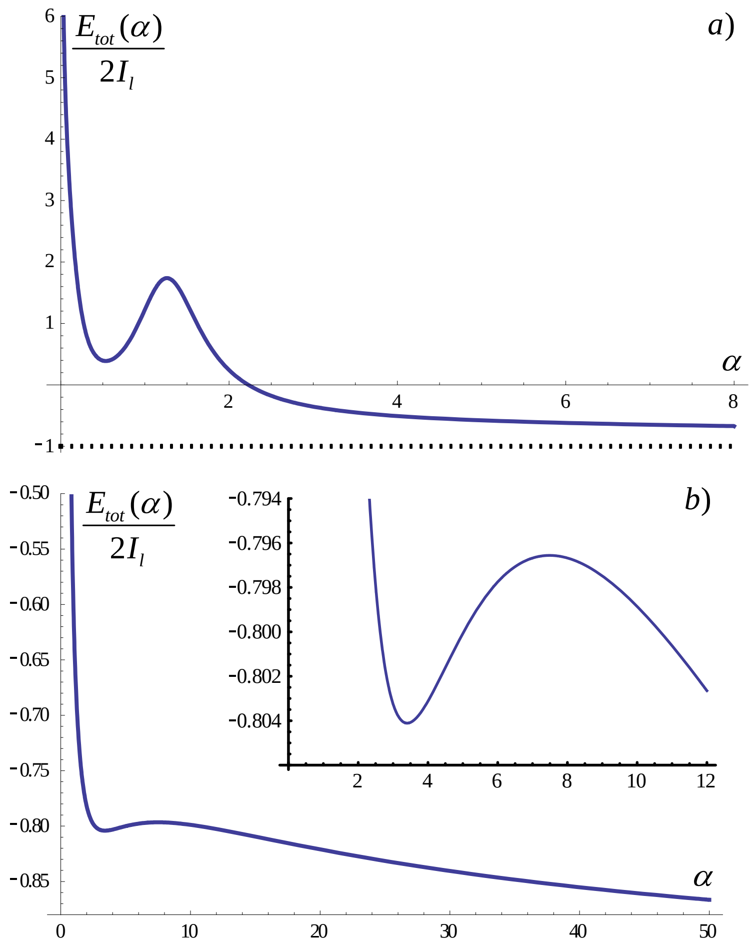

Fig. 4. Total energies of two bound 2D magnetoexcitons with wave vectors $\vec{k}$ and $-\vec{k}$, with different spin structures $\eta = \pm 1$ and with trial wave function $\varphi_2(k)$, as the functions of parameter α. (a) the case of $\eta = 1$, (b) the case of $\eta = -1$. The total energies are related to the value of $2I_l$, where I_l is the ionization potential of a free magnetoexciton with wave vector $k = 0$.

The numerical calculations effectuated in the case of the function $\varphi_2(\vec{k}) = (8\alpha^3)^{1/2}(kl_0)^2 e^{-\alpha(kl_0)^2}$ made it possible to obtain the full energies of the bound states as a function of parameter α of the trial wave function in two cases with $\eta = \pm 1$, corresponding to two electron and hole spin structures. In both spin configurations, the full energies of the bound states are greater than the value of $-2I_l$ in the entire range of α. All these states are unstable as regards the dissociation in the form of two free magnetoexcitons with $\vec{k} = 0$. Despite this fact, a deep metastable bound state with the activation barrier comparable to two magnetoexciton ionization potentials $2I_l$ was revealed in the case of $\eta = 1$ and $\alpha = 0.5$. Only a shallow state was found in the case of $\eta = -1$ and $\alpha = 3.4$.

5. Conclusions

In the LLL approximation, when the influence of the ELLs is neglected, the bound states of two magnetoexcitons with opposite wave vectors $\vec{k}$ and $-\vec{k}$ have been studied. The electrons and the heavy holes are situated on the LLLs with the cyclotron energies greater than the binding energy of the 2D Wannier–Mott exciton. The spin states of two electrons and the effective spin states of two heavy holes have been combined so as to form symmetric or antisymmetric states separately, characterized by parameter $\eta = \pm 1$. Each magnetoexciton with wave vector $\vec{k} \neq 0$ looks as an electric dipole with the length of the arm between the electron and the hole equal to $d = kl_0^2$, where l_0 is the magnetic length. The arm is oriented in-plane perpendicular to wave vector $\vec{k}$. It has been supposed that the bound states are formed by two dipoles with antiparallel arms and wave vectors, yet with the total wave vector equal to zero. The bound pair of two dipoles is oriented arbitrarily in the plane of the layer and characterized by the trial wave function of relative motion $\varphi_n(\vec{k})$, which depends only on modulus $|\vec{k}|$. The numerical calculations with trial wave function $\varphi_2(k) = (8\alpha^3)^{1/2}(kl_0)^2 e^{-\alpha(kl_0)^2}$ have demonstrated the absence of stable molecular bound states in both spin orientations $\eta = \pm 1$. Instead of them, the metastable bound states have been revealed. One of them is a deep bound state. It does exist in the case of $\eta = 1$ and $\alpha = 0.5$ and is characterized by the activation barrier comparable with two magnetoexciton ionization potentials $2I_l$. In the case of $\eta = -1$ and $\alpha = 3.4$, only a shallow metastable bound state can be formed.

Appendix

The exact values of integrals I_8 (36) and I_9 (37) are given below.

$$I_8=\Bigg[\Bigg(\Bigg(-\sqrt{\frac{\pi}{2}}\Bigg(\frac{32a^2}{(1+2a)^2(1+4a^2)^3}+\frac{16a}{(1+2a)^3(1+4a^2)^2}-$$

$$-\frac{4}{(1+2a)^2(1+4a^2)^2}+\frac{4}{(1+2a)^4(1+4a^2)}\Bigg)\Bigg/\Bigg(2\sqrt{\frac{1}{(1+2a)^2(1+4a^2)}}\Bigg)+$$

$$+\frac{\sqrt{\frac{\pi}{2}}\left(-\frac{4a}{(1+2a)^2(1+4a^2)^2}-\frac{2}{(1+2a)^3(1+4a^2)}\right)^2}{4\left(\frac{1}{(1+2a)^2(1+4a^2)}\right)^{3/2}}\Bigg)\times{}_2F_1[1/4,3/4,1,-\frac{4}{(1+2a)^4}]+$$

$$+\left(-\frac{3\left(-\frac{4a}{(1+2a)^2(1+4a^2)^2}-\frac{2}{(1+2a)^3(1+4a^2)}\right)\sqrt{\frac{\pi}{2}}}{2(1+2a)^5\sqrt{\frac{1}{(1+2a)^2(1+4a^2)}}}\right)\times{}_2F_1[5/4,7/4,2,-\frac{4}{(1+2a)^4}]\Bigg]+$$

$$\Bigg((-\frac{3\left(-\frac{4a}{(1+2a)^2(1+4a^2)^2}-\frac{2}{(1+2a)^3(1+4a^2)}\right)\sqrt{\frac{\pi}{2}}}{2(1+2a)^5\sqrt{\frac{1}{(1+2a)^2(1+4a^2)}}}+$$

$$+\frac{6\sqrt{\frac{1}{(1+2a)^2(1+4a^2)}}\sqrt{2\pi}}{(1+2a)^6})\times{}_2F_1[5/4,7/4,2,-\frac{4}{(1+2a)^4}]+ \tag{A1}$$

$$+\left(-\frac{105\sqrt{\frac{1}{(1+2a)^2(1+4a^2)}}\sqrt{\frac{\pi}{2}}}{2(1+2a)^{10}}\right)\times{}_2F_1[9/4,11/4,3,-\frac{4}{(1+2a)^4}]\Bigg);$$

Integral I_9 is a series expansion, whose terms are triple products of the Bessel functions of the orders $2n$ with $n = 1, 2, \ldots, \infty$. The first ten terms of the sum with n=1, 2, …, 10 were taken into account explicitly. The negligible contributions of the next terms with $n \geq 11$ have been verified carefully.

$$I_9 = 2\sqrt{2}(\frac{1}{1+4a^2})^{3/2} \times (\frac{3\sqrt{\pi}\,{}_2F_1[\frac{5}{4},\frac{7}{4},3,-\frac{4}{(1+2a)^4}]}{8((1+2a)^2)^{5/2}} +$$

$$\frac{35\sqrt{\pi}\,{}_2F_1[\frac{9}{4},\frac{11}{4},5,-\frac{4}{(1+2a)^4}]}{128((1+2a)^2)^{9/2}} + \frac{231\sqrt{\pi}\,{}_2F_1[\frac{13}{4},\frac{15}{4},7,-\frac{4}{(1+2a)^4}]}{1024((1+2a)^2)^{13/2}} +$$

$$\frac{6435\sqrt{\pi}\,{}_2F_1[\frac{17}{4},\frac{19}{4},9,-\frac{4}{(1+2a)^4}]}{32768((1+2a)^2)^{17/2}} + \frac{46189\sqrt{\pi}\,{}_2F_1[\frac{21}{4},\frac{23}{4},11,-\frac{4}{(1+2a)^4}]}{262144((1+2a)^2)^{21/2}} +$$

$$\frac{676039\sqrt{\pi}\,{}_2F_1[\frac{25}{4},\frac{27}{4},13,-\frac{4}{(1+2a)^4}]}{4194304((1+2a)^2)^{25/2}} + \frac{5014575\sqrt{\pi}\,{}_2F_1[\frac{29}{4},\frac{31}{4},15,-\frac{4}{(1+2a)^4}]}{33554432((1+2a)^2)^{29/2}} +$$

$$\frac{300540195\sqrt{\pi}\,{}_2F_1[\frac{33}{4},\frac{35}{4},17,-\frac{4}{(1+2a)^4}]}{2147483648((1+2a)^2)^{33/2}} + \frac{2268783825\sqrt{\pi}\,{}_2F_1[\frac{37}{4},\frac{39}{4},19,-\frac{4}{(1+2a)^4}]}{17179869184((1+2a)^2)^{37/2}} +$$

$$(34461632205\sqrt{\pi}\,{}_2F_1[\frac{41}{4},\frac{43}{4},21,-\frac{4}{(1+2a)^4}])/(274877906944((1+2a)^2)^{41/2})) -$$

$$12\sqrt{2}a^2(\frac{1}{1+4a^2})^{5/2} \times (\frac{3\sqrt{\pi}\,{}_2F_1[\frac{5}{4},\frac{7}{4},3,-\frac{4}{(1+2a)^4}]}{8((1+2a)^2)^{5/2}} +$$

$$\frac{35\sqrt{\pi}\,{}_2F_1[\frac{9}{4},\frac{11}{4},5,-\frac{4}{(1+2a)^4}]}{128((1+2a)^2)^{9/2}} + \frac{231\sqrt{\pi}\,{}_2F_1[\frac{13}{4},\frac{15}{4},7,-\frac{4}{(1+2a)^4}]}{1024((1+2a)^2)^{13/2}} +$$

$$\frac{6435\sqrt{\pi}\,{}_2F_1[\frac{17}{4},\frac{19}{4},9,-\frac{4}{(1+2a)^4}]}{32768((1+2a)^2)^{17/2}} + \frac{46189\sqrt{\pi}\,{}_2F_1[\frac{21}{4},\frac{23}{4},11,-\frac{4}{(1+2a)^4}]}{262144((1+2a)^2)^{21/2}} +$$

$$\frac{676039\sqrt{\pi}\,{}_2F_1[\frac{25}{4},\frac{27}{4},13,-\frac{4}{(1+2a)^4}]}{4194304((1+2a)^2)^{25/2}} + \frac{5014575\sqrt{\pi}\,{}_2F_1[\frac{29}{4},\frac{31}{4},15,-\frac{4}{(1+2a)^4}]}{33554432((1+2a)^2)^{29/2}} +$$

$$\frac{300540195\sqrt{\pi}\,{}_2F_1[\frac{33}{4},\frac{35}{4},17,-\frac{4}{(1+2a)^4}]}{2147483648((1+2a)^2)^{33/2}} + \frac{2268783825\sqrt{\pi}\,{}_2F_1[\frac{37}{4},\frac{39}{4},19,-\frac{4}{(1+2a)^4}]}{17179869184((1+2a)^2)^{37/2}} +$$

$$(34461632205\sqrt{\pi}\,{}_2F_1[\frac{41}{4},\frac{43}{4},21,-\frac{4}{(1+2a)^4}])/(274877906944((1+2a)^2)^{41/2})) +$$

$$4\sqrt{2}a(\frac{1}{1+4a^2})^{3/2}\times(-\frac{15(1+2a)\sqrt{\pi}\,{}_2F_1[\frac{5}{4},\frac{7}{4},3,-\frac{4}{(1+2a)^4}]}{8((1+2a)^2)^{7/2}}+$$

$$\frac{35\sqrt{\pi}\,{}_2F_1[\frac{9}{4},\frac{11}{4},4,-\frac{4}{(1+2a)^4}]}{8(1+2a)^5((1+2a)^2)^{5/2}}-\frac{315(1+2a)\sqrt{\pi}\,{}_2F_1[\frac{9}{4},\frac{11}{4},5,-\frac{4}{(1+2a)^4}]}{128((1+2a)^2)^{11/2}}+$$

$$\frac{693\sqrt{\pi}\,{}_2F_1[\frac{13}{4},\frac{15}{4},6,-\frac{4}{(1+2a)^4}]}{128(1+2a)^5((1+2a)^2)^{9/2}}-\frac{3003(1+2a)\sqrt{\pi}\,{}_2F_1[\frac{13}{4},\frac{15}{4},7,-\frac{4}{(1+2a)^4}]}{1024((1+2a)^2)^{15/2}}+$$

$$\frac{6435\sqrt{\pi}\,{}_2F_1[\frac{17}{4},\frac{19}{4},8,-\frac{4}{(1+2a)^4}]}{1024(1+2a)^5((1+2a)^2)^{13/2}}-$$

$$(109395(1+2a)\sqrt{\pi}\,{}_2F_1[\frac{17}{4},\frac{19}{4},9,-\frac{4}{(1+2a)^4}])/(32768((1+2a)^2)^{19/2})+$$

$$\frac{230945\sqrt{\pi}\,{}_2F_1[\frac{21}{4},\frac{23}{4},10,-\frac{4}{(1+2a)^4}]}{32768(1+2a)^5((1+2a)^2)^{17/2}}-$$

$$(969969(1+2a)\sqrt{\pi}\,{}_2F_1[\frac{21}{4},\frac{23}{4},11,-\frac{4}{(1+2a)^4}])/(262144((1+2a)^2)^{23/2})+$$

$$\frac{2028117\sqrt{\pi}\,{}_2F_1[\frac{25}{4},\frac{27}{4},12,-\frac{4}{(1+2a)^4}]}{262144(1+2a)^5((1+2a)^2)^{21/2}}-$$

$$(16900975(1+2a)\sqrt{\pi}\,{}_2F_1[\frac{25}{4},\frac{27}{4},13,-\frac{4}{(1+2a)^4}])/(4194304((1+2a)^2)^{27/2})+$$

$$\frac{35102025\sqrt{\pi}\,{}_2F_1[\frac{29}{4},\frac{31}{4},14,-\frac{4}{(1+2a)^4}]}{4194304(1+2a)^5((1+2a)^2)^{25/2}}-$$

$$(145422675(1+2a)\sqrt{\pi}\,{}_2F_1[\frac{29}{4},\frac{31}{4},15,-\frac{4}{(1+2a)^4}])/(33554432((1+2a)^2)^{31/2})+$$

$$\frac{300540195\sqrt{\pi}\,{}_2F_1[\frac{33}{4},\frac{35}{4},16,-\frac{4}{(1+2a)^4}]}{33554432(1+2a)^5((1+2a)^2)^{29/2}}-$$

$$(9917826435(1+2a)\sqrt{\pi}\,{}_2F_1[\frac{33}{4},\frac{35}{4},17,-\frac{4}{(1+2a)^4}])/(2147483648((1+2a)^2)^{35/2})+$$

$$(20419054425\sqrt{\pi}\,{}_2F_1[\frac{37}{4},\frac{39}{4},18,-\frac{4}{(1+2a)^4}])/(2147483648(1+2a)^5((1+2a)^2)^{33/2})-$$

$$(83945001525(1+2a)\sqrt{\pi}\,{}_2F_1[\frac{37}{4},\frac{39}{4},19,-\frac{4}{(1+2a)^4}])/(17179869184((1+2a)^2)^{39/2})+$$

$$(172308161025\sqrt{\pi}\,{}_2F_1[\frac{41}{4},\frac{43}{4},20,-\frac{4}{(1+2a)^4}])/(17179869184(1+2a)^5((1+2a)^2)^{37/2})-$$

$$(1412926920405(1+2a)\sqrt{\pi}\,{}_2F_1[\frac{41}{4},\frac{43}{4},21,-\frac{4}{(1+2a)^4}])/(274877906944((1+2a)^2)^{43/2})+$$

$$(2893136075115\sqrt{\pi}\,{}_2F_1[\frac{45}{4},\frac{47}{4},22,-\frac{4}{(1+2a)^4}])/(274877906944(1+2a)^5((1+2a)^2)^{41/2}))-$$

$$\sqrt{2}\sqrt{\frac{1}{1+4a^2}}\Bigg(\frac{75\sqrt{\pi}\,{}_2F_1[\frac{5}{4},\frac{7}{4},3,-\frac{4}{(1+2a)^4}]}{8((1+2a)^2)^{7/2}}-$$

$$\frac{175\sqrt{\pi}\,{}_2F_1[\frac{9}{4},\frac{11}{4},4,-\frac{4}{(1+2a)^4}]}{4(1+2a)^4((1+2a)^2)^{7/2}}-\frac{35\sqrt{\pi}\,{}_2F_1[\frac{9}{4},\frac{11}{4},4,-\frac{4}{(1+2a)^4}]}{2(1+2a)^6((1+2a)^2)^{5/2}}+$$

$$\frac{2835\sqrt{\pi}\,{}_2F_1[\frac{9}{4},\frac{11}{4},5,-\frac{4}{(1+2a)^4}]}{128((1+2a)^2)^{11/2}}+\frac{3465\sqrt{\pi}\,{}_2F_1[\frac{13}{4},\frac{15}{4},5,-\frac{4}{(1+2a)^4}]}{32(1+2a)^{10}((1+2a)^2)^{5/2}}-$$

$$\frac{6237\sqrt{\pi}\,{}_2F_1[\frac{13}{4},\frac{15}{4},6,-\frac{4}{(1+2a)^4}]}{64(1+2a)^4((1+2a)^2)^{11/2}}-\frac{693\sqrt{\pi}\,{}_2F_1[\frac{13}{4},\frac{15}{4},6,-\frac{4}{(1+2a)^4}]}{32(1+2a)^6((1+2a)^2)^{9/2}}+$$

$$\frac{39039\sqrt{\pi}\,{}_2F_1[\frac{13}{4},\frac{15}{4},7,-\frac{4}{(1+2a)^4}]}{1024((1+2a)^2)^{15/2}}+\frac{45045\sqrt{\pi}\,{}_2F_1[\frac{17}{4},\frac{19}{4},7,-\frac{4}{(1+2a)^4}]}{256(1+2a)^{10}((1+2a)^2)^{9/2}}-$$

$$\frac{83655\sqrt{\pi}\,{}_2F_1[\frac{17}{4},\frac{19}{4},8,-\frac{4}{(1+2a)^4}]}{512(1+2a)^4((1+2a)^2)^{15/2}}-$$

$$\frac{6435\sqrt{\pi}\,{}_2F_1[\frac{17}{4},\frac{19}{4},8,-\frac{4}{(1+2a)^4}]}{256(1+2a)^6((1+2a)^2)^{13/2}}+\frac{1859715\sqrt{\pi}\,{}_2F_1[\frac{17}{4},\frac{19}{4},9,-\frac{4}{(1+2a)^4}]}{32768((1+2a)^2)^{19/2}}+$$

$$\frac{2078505\sqrt{\pi}\,{}_2F_1[\frac{21}{4},\frac{23}{4},9,-\frac{4}{(1+2a)^4}]}{8192(1+2a)^{10}((1+2a)^2)^{13/2}}-\frac{3926065\sqrt{\pi}\,{}_2F_1[\frac{21}{4},\frac{23}{4},10,-\frac{4}{(1+2a)^4}]}{16384(1+2a)^4((1+2a)^2)^{19/2}}-$$

$$\frac{230945\sqrt{\pi}\,{}_2F_1[\frac{21}{4},\frac{23}{4},10,-\frac{4}{(1+2a)^4}]}{8192(1+2a)^6((1+2a)^2)^{17/2}}+$$

$$\frac{20369349\sqrt{\pi}\,{}_2F_1[\frac{21}{4},\frac{23}{4},11,-\frac{4}{(1+2a)^4}]}{262144((1+2a)^2)^{23/2}}+\frac{22309287\sqrt{\pi}\,{}_2F_1[\frac{25}{4},\frac{27}{4},11,-\frac{4}{(1+2a)^4}]}{65536(1+2a)^{10}((1+2a)^2)^{17/2}}-$$

$$\frac{42590457\sqrt{\pi}\,{}_2F_1[\frac{25}{4},\frac{27}{4},12,-\frac{4}{(1+2a)^4}]}{131072(1+2a)^4((1+2a)^2)^{23/2}}-\frac{2028117\sqrt{\pi}\,{}_2F_1[\frac{25}{4},\frac{27}{4},12,-\frac{4}{(1+2a)^4}]}{65536(1+2a)^6((1+2a)^2)^{21/2}}+$$

$$\frac{422524375\sqrt{\pi}\,{}_2F_1[\frac{25}{4},\frac{27}{4},13,-\frac{4}{(1+2a)^4}]}{4194304((1+2a)^2)^{27/2}}+$$

$$\frac{456326325\sqrt{\pi}\,{}_2F_1[\frac{29}{4},\frac{31}{4},13,-\frac{4}{(1+2a)^4}]}{1048576(1+2a)^{10}((1+2a)^2)^{21/2}}-\frac{877550625\sqrt{\pi}\,{}_2F_1[\frac{29}{4},\frac{31}{4},14,-\frac{4}{(1+2a)^4}]}{2097152(1+2a)^4((1+2a)^2)^{27/2}}-$$

$$\frac{35102025\sqrt{\pi}\,{}_2F_1[\frac{29}{4},\frac{31}{4},14,-\frac{4}{(1+2a)^4}]}{1048576(1+2a)^6((1+2a)^2)^{25/2}}+\frac{4217257575\sqrt{\pi}\,{}_2F_1[\frac{29}{4},\frac{31}{4},15,-\frac{4}{(1+2a)^4}]}{33554432((1+2a)^2)^{31/2}}+$$

$$\frac{4508102925\sqrt{\pi}\,{}_2F_1[\frac{33}{4},\frac{35}{4},15,-\frac{4}{(1+2a)^4}]}{8388608(1+2a)^{10}((1+2a)^2)^{25/2}}-$$

$$\frac{8715665655\sqrt{\pi}\,{}_2F_1[\frac{33}{4},\frac{35}{4},16,-\frac{4}{(1+2a)^4}]}{16777216(1+2a)^4((1+2a)^2)^{31/2}}-\frac{300540195\sqrt{\pi}\,{}_2F_1[\frac{33}{4},\frac{35}{4},16,-\frac{4}{(1+2a)^4}]}{8388608(1+2a)^6((1+2a)^2)^{29/2}}+$$

$$(327288272355\sqrt{\pi}\,{}_2F_1[\frac{33}{4},\frac{35}{4},17,-\frac{4}{(1+2a)^4}])/(2147483648((1+2a)^2)^{35/2})+$$

$$(347123925225\sqrt{\pi}\,{}_2F_1[\frac{37}{4},\frac{39}{4},17,-\frac{4}{(1+2a)^4}])/(536870912(1+2a)^{10}((1+2a)^2)^{29/2})-$$

$$(673828796025\sqrt{\pi}\,{}_2F_1[\frac{37}{4},\frac{39}{4},18,-\frac{4}{(1+2a)^4}])/(1073741824(1+2a)^4((1+2a)^2)^{35/2})-$$

$$(20419054425\sqrt{\pi}\,{}_2F_1[\frac{37}{4},\frac{39}{4},18,-\frac{4}{(1+2a)^4}])/(536870912(1+2a)^6((1+2a)^2)^{33/2})+$$

$$(3105965056425\sqrt{\pi}\,{}_2F_1[\frac{37}{4},\frac{39}{4},19,-\frac{4}{(1+2a)^4}])/(17179869184((1+2a)^2)^{39/2})+$$

$$(3273855059475\sqrt{\pi}\,{}_2F_1[\frac{41}{4},\frac{43}{4},19,-\frac{4}{(1+2a)^4}])/(4294967296(1+2a)^{10}((1+2a)^2)^{33/2})-$$

$$(6375401957925\sqrt{\pi}\,{}_2F_1[\frac{41}{4},\frac{43}{4},20,-\frac{4}{(1+2a)^4}])/(8589934592(1+2a)^4((1+2a)^2)^{39/2})-$$

$$(172308161025\sqrt{\pi}\,{}_2F_1[\frac{41}{4},\frac{43}{4},20,-\frac{4}{(1+2a)^4}])/(4294967296(1+2a)^6((1+2a)^2)^{37/2})+$$

$$(57930003736605\sqrt{\pi}\times$$
$$_2F_1[\frac{41}{4},\frac{43}{4},21,-\frac{4}{(1+2a)^4}])/(274877906944((1+2a)^2)^{43/2})+$$
$$(60755857577415\sqrt{\pi}\times$$
$$_2F_1[\frac{45}{4},\frac{47}{4},21,-\frac{4}{(1+2a)^4}])/(68719476736(1+2a)^{10}((1+2a)^2)^{37/2})-$$
$$(118618579079715\sqrt{\pi}\times$$
$$_2F_1[\frac{45}{4},\frac{47}{4},22,-\frac{4}{(1+2a)^4}])/(137438953472(1+2a)^4((1+2a)^2)^{43/2})-$$
$$(2893136075115\sqrt{\pi}\times$$
$$_2F_1[\frac{45}{4},\frac{47}{4},22,-\frac{4}{(1+2a)^4}])/(68719476736(1+2a)^6((1+2a)^2)^{41/2})+$$
$$(556271163533475\sqrt{\pi}\times$$
$$_2F_1[\frac{49}{4},\frac{51}{4},23,-\frac{4}{(1+2a)^4}])/(549755813888(1+2a)^{10}((1+2a)^2)^{41/2})). \tag{A2}$$

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