COMMENTS ON ACOUSTIC WAVE PROPAGATION IN STRATIFIED MEDIA

S. Cojocaru

Department of Theoretical Physics,
Horia Hulubei National Institute for Physics and Nuclear Engineering,
077125 Magurele, Romania
E-mail: scojocaru@theory.nipne.ro

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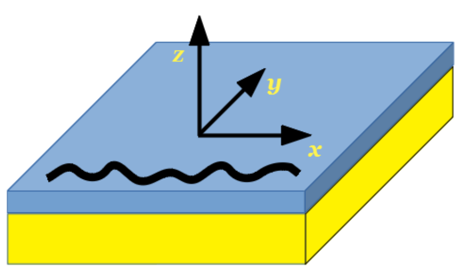
Abstract

In a layered system, the interface between two media presents an intrinsic inhomogeneity even when the materials are isotropic and homogeneous in the sense of continuum medium description of wave propagation. In terms of the rigid bonding (RB) model, the material parameters behave as piecewise continuous functions with an abrupt change across the interface. Therefore, it is reasonable to expect that the wave experiences a singular potential at the border formed by the space derivatives of the material parameters. However, we prove that this potential is strictly cancelled due to continuity of the stress field. The equations governing acoustic wave propagation are derived by the partial wave decomposition method.

1. Introduction

Wave propagation in composite elastic media is a topic of major interest with a large spectrum of applications in various areas of physics spanning a large scale from geophysics [1] to nanophysics [2–4]. In relation to this topic, a large number of theoretical models, methods and approaches have been developed to describe the situation in terms of continuum medium (see, e.g., [5–11]).



One of the most general methods uses the decomposition of displacement field vector $\boldsymbol{U}(\boldsymbol{r},t)$ into partial waves (point $\boldsymbol{r}=(x,y,z)$ at time t). For the case of a composite slab or plate geometry (Fig. 1), translation symmetry is broken in one of the space directions (z in the figure) so that natural modes of vibration are formed due to multiple wave reflections and refractions. Translation symmetry is assumed to hold in the other two directions; therefore, the decomposition can be represented as follows:

$$\mathbf{U}(\mathbf{r},t) = \sum_n \int \mathbf{u}_n(z,q,\omega_n)\exp(iqx - i\omega_n t)\frac{dq}{(2\pi)}. \tag{1}$$

The x axis is chosen along the direction of wave propagation; index n of the branches in the frequency dispersion spectrum $\omega(q)$ is further dropped. The interface is at $z = 0$; the outer surfaces correspond to $z = d_a$ and $z = -d_b$, so that the total thickness is $d = d_a + d_b$. Considering the wave reflection from the surfaces [12], it can be shown that the spectrum decouples into shear horizontal waves $u(z) = (0,u_y(z),0)$ and Rayleigh–Lamb modes with a mixed shear vertical (s) and longitudinal (l) polarization $u(z) = (u_x(z),0,u_z(z))$. Below, we consider these last ones for definiteness.

2. Equation for the Rayleigh–Lamb Waves

According to the linear elasticity theory, displacement field vector $\boldsymbol{U}(\boldsymbol{r},t)$ is linearly related to the strain tensor

$$\varepsilon_{ij} = \left(\frac{\partial U_i}{\partial r_j} + \frac{\partial U_j}{\partial r_i}\right)/2. \tag{2}$$

For an isotropic medium, stiffness tensor λ_{ijkm} linking the strain to stress tensor $\Sigma_{ij}(\mathbf{r},t)$ simplifies to $\lambda_{ijkm} = \lambda\delta_{ij}\delta_{km} + \mu(\delta_{ik}\delta_{jm}+\delta_{im}\delta_{jk})$, where λ and μ are the two Lamé parameters and δ_{ij} is the Kronecker δ-symbol with $\{i,j,k,m\} = x,y,z$. Then the constitutive equation takes the form

$$\Sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk}+2\mu\varepsilon_{ij}. \tag{3}$$

The equation of e lastodynamics essentially derives from the Newton's second law

$$\rho\frac{\partial^2 U_i(\mathbf{r},t)}{\partial t^2} = \frac{\partial \Sigma_{ij}(\mathbf{r},t)}{\partial r_j}. \tag{4}$$

Here, ρ is the mass density. The partial wave decomposition of the stress tensor corresponding to (1) is as follows:

$$\Sigma_{ij}(\mathbf{r},t) = \sum_n \int \sigma_{ij}(z)\exp(iqx - i\omega_n t)\frac{dq}{(2\pi)}. \tag{5}$$

The traction free boundary conditions and the continuity on the interface provide eight equations for the unknown amplitudes and frequency ω:

$$\begin{aligned} u_{x,a}(z=0) &= u_{x,b}(z=0), \quad u_{z,a}(z=0) = u_{z,b}(z=0), \\ \sigma_{zx,a}(z=0) &= \sigma_{zx,b}(z=0), \quad \sigma_{zz,a}(z=0) = \sigma_{zz,b}(z=0), \end{aligned}$$

$$\begin{aligned} \sigma_{xz,a}(z=d_a) &= 0, \; \sigma_{xz,b}(z=-d_b) = 0, \\ \sigma_{zz,a}(z=d_a) &= 0, \; \sigma_{zz,b}(z=-d_b) = 0. \end{aligned} \tag{6}$$

The “trouble” with this model is that the elastic constants in (3) are actually not constant at the interface where they change abruptly from one material to the other, while in the published literature this variation is usually ignored [7–11]. On general grounds, one expects that near the interface, the material parameters would have a smooth variation, which can be modeled either dynamically via springs or by specifying the space variation of all the material parameters [4–6, 13]. This leads to more complicated equations than the ones usually considered in the model with rigidly bonded layers. It is clear that the steeper the variation in, e.g., $\lambda(z)$ and $\mu(z)$, the more significant the effect of the respective derivative terms $d\lambda(z)/dz$, $d\mu(z)/dz$ in (4). This argument appears to be in contrast to the treatment of the rigid bonding (RB) limit mentioned before.

The continuity conditions in (6) allow representing the z-functions in (1) and (5) as follows:

$$\begin{aligned} \sigma_{xz}(z) &= \theta(z)\, \sigma_{xz,a}(z) + (1-\theta(z))\, \sigma_{xz,b}(z), \\ \sigma_{zz}(z) &= \theta(z)\, \sigma_{zz,a}(z) + (1-\theta(z))\, \sigma_{zz,b}(z), \\ u_i(z) &= \theta(z)\, u_{i,a}(z) + (1-\theta(z))\, u_{i,b}(z). \end{aligned} \tag{7}$$

The Heaviside θ-function has the known property $d\theta(z)/dz = \delta(z)$. After substitution of the above expressions into (4), we obtrain

$$\begin{aligned} \rho\frac{\partial^2 U_x}{\partial t^2} &= \sum_n \int \left(iq\sigma_{xx}(z) + \frac{\partial \sigma_{xz}}{\partial z}\right) \exp\left(iqx - i\omega t\right) \frac{dq}{2\pi}, \\ \rho\frac{\partial^2 U_z}{\partial t^2} &= \sum_n \int \left(iq\sigma_{xz}(z) + \frac{\partial \sigma_{zz}}{\partial z}\right) \exp\left(iqx - i\omega t\right) \frac{dq}{2\pi}. \end{aligned} \tag{8}$$

The contribution of the derivatives in the right-hand-side of the above expressions contains terms with Dirac delta-function

$$\delta(z)(\sigma_{ij,a}(z) - \sigma_{ij,b}(z), \tag{9}$$

which vanish under the continuity condition. It should be noted that when equations (6) are solved numerically, e.g., by finite difference methods, this cancellation can be easily “miscalculated”. All the remaining terms have either $\theta(z)$ or $(1-\theta(z))$ as a prefactor and, by definition, refer to one of the two layers where the material parameters are constants. In addition, note that generally $\sigma_{xx}(z)$, as well as $\sigma_{yy}(z)$, are not continuous functions of z: by definition, the respective tensor components represent forces acting normally to the surfaces, respectively, x = const and y = const in the figure; that is, they are parallel to the interface and, therefore, counterbalanced by an equal and opposite internal force within each layer by the Newton’s third law. Therefore, the values of these forces on the two sides of the interface can differ; they can be written in the form with the θ-function similar to (7).

From (2), (3), and (5) we obtain the explicit expressions

$$\begin{aligned} \sigma_{xx} &= \left(\lambda \frac{\partial u_z}{\partial z} + (\lambda + 2\mu)\,(iq)\,u_x\right), \\ \sigma_{xz} &= \mu\left(\frac{\partial u_x}{\partial z} + iqu_z\right), \\ \sigma_{zz} &= \lambda iqu_x + (\lambda + 2\mu)\,\frac{\partial u_z}{\partial z}. \end{aligned} \tag{10}$$

It is evident that these equations are consistent with representation (7) by virtue of property (8), where stresses are replaced by displacement amplitudes. By taking the remaining derivatives in (4), we recover the commonly used form of the elastodynamic equations describing Rayleigh–Lamb waves in stratified isotropic media in terms of the RB model, e.g., [7–10]:

$$\begin{aligned} &\mu_\gamma \frac{\partial^2 u_{x,\gamma}}{\partial z^2} - (\lambda_\gamma + 2\mu_\gamma) q^2 w_\gamma^2 u_{x,\gamma} + (\lambda_\gamma + \mu_\gamma) q \frac{\partial i u_{z,\gamma}}{\partial z} = 0, \\ &(\lambda_\gamma + 2\mu_\gamma) \frac{\partial^2 i u_{z,\gamma}}{\partial z^2} - \mu_\gamma q^2 v_\gamma^2 i u_{z,\gamma} - (\lambda_\gamma + \mu_\gamma) q \frac{\partial u_{x,\gamma}}{\partial z} = 0, \end{aligned} \tag{11}$$

where γ = a,b and the quantities

$$w_\gamma = \sqrt{1 - \left(\frac{\omega}{q\ell_\gamma}\right)^2}, \quad v_\gamma = \sqrt{1 - \left(\frac{\omega}{qs_\gamma}\right)^2}, \tag{12}$$

contain the normal mode frequencies scaled with the phase velocities of the bulk longitudinal and transverse sound waves for each of the materials:

$$\ell = \sqrt{(\lambda + 2\mu)/\rho}, \quad s = \sqrt{\mu/\rho}. \tag{13}$$

Boundary conditions (6), together with the normalization condition

$$\int_{-d_b}^{d_a} |u_n(z)|^2 dz = 1, \tag{14}$$

completely determine the wave propagation spectrum in the layered system.

3. Conclusions

One of the basic assumptions of the continuum description of the elastic wave propagation is that the wavelength is significantly larger than the atomic scale variation of the material properties. It is therefore justifiable to use the RB model with an abrupt change in material parameters, provided the interface of a layered system occupies a few atomic layers. In this case, the singular terms originating from the derivatives of the elastic parameters at the boundary do not produce any effect on the wave propagation and can be safely ignored. The physical reason for this "annihilation of singularity" effect is the continuity of stresses or forces acting on the interface.

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