

NONLINEAR HAMILTONIAN DYNAMICS OF COHERENT PHOTONS AND PHONONS IN BIOLOGICAL MEDIA

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Abstract

Appearance of ultrashort dynamic chaos of phonons was investigated under the action of the external electromagnetic field in biological media. The system of differential equations that describes the system evolution and the numeric solutions of the system were obtained. Different regimes of oscillations were observed to occur depending on the system parameters. The parameter values were determined at which ultrashort dynamic chaos can be observed.

1. Introduction

Recently, much attention in various fields of nature science has been paid to the problem of self organization of various temporal, special, special-temporal, and functional structures.

Nonlinear interaction of light and matter, semiconductors and dielectrics inclusive, is one of the most important problems of modern physics.

Development of powerful laser radiation sources in a temporal interval opens new directions of investigation of light interaction with matter, depending on interaction nature, ratio between temporal duration of laser pulse and characteristic relaxation the appearance of diverse nonlinear cooperative phenomena. Among the most important ones, there are super radiation, super fluorescence, photonic echo, self-induced transparency, optical bistability and multistability, turbulence, optical dynamic chaos, etc. One could mention that dynamic chaos discovery, generally, is considered one of the greatest sensations of modern science. In the last 15-20 years, the determinist chaos, both in classical and quantum systems, has evolved into a distinct science named chaosology.

An extremely important domain is studying of dynamic chaos in nonlinear optical systems, both dissipative and Hamiltonian. In the last case the phenomena and processes evolve in temporal periods less than the characteristic relaxation time.

The investigation of optical self-organization at resonant excitation of excitons is of a special interest due to the giant nonlinearities.

Creation of powerful sources of light radiation allowed growing considerable exciton concentration, so the average distance between excitons is comparable with their radius. In such conditions, at low temperatures, essentially new cooperative phenomena of excitons are shown due to interaction between them. Multifarious phenomena and effects could appear depending on interaction character. S. Moskalenko [1], N. Lampert [2], and R. Merryfield [3] have predicted theoretically the possibility of formation of an exciton molecule - biexciton - in a crystal in the case when excitons attract each other. Further, the biexciton was discovered experimentally, and actually the physics of biexcitons became a distinct discipline - biexcitonics.

Another vivid display is considered Bose-Einstein condensation of excitons when excitons repel each other. This phenomenon was predicted by S. Moskalenko [4-5].

At low temperatures a phase transition and formation of metallic drop are possible in large radius exciton systems due to attraction between them. This phenomenon was predicted by L. Keldysh [6].

Numerous articles and monographs are concerned with physics of high-density excitons. The current stage of physics of excitons and biexcitons is determined by rapid development of nonlinear physics, synergetics, phase-transition physics, laser physics, physics of electronic monostructures, etc. [7-10].

The character of nonlinear effects in phononic range of spectrum differs significantly from the effects that take place at coherent interaction of laser radiation with atoms. The excitation of the crystal atom under the influence of laser radiation is transmitted to the neighbor atoms quicker than the atom mutation occurs. This particularity leads to impossibility to assume the results obtained at nonlinear coherent interaction of laser radiation with the two-level atomic systems at studying analogical phenomena in the phonon spectrum range. The processes of self-organization, stability, disintegration of diverse optic structures biological media, phononic and biphononic ones inclusive, have particular proprieties. Relevance of these investigations is determined, first of all, by necessity to study the formation of crucially new structures in biological media, to predict new phenomena, and to use them for design of new optoelectronic devices.

2. Model Hamiltonian and equation of motion

Nonlinear dynamics of coherent phonons and photons with account of non-resonant terms is studied. The system model is represented in Fig. 1.

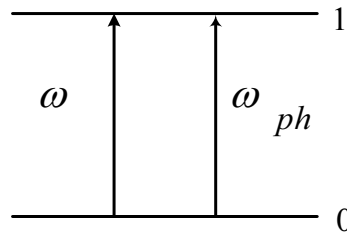


Fig. 1. Energy level diagram of one-photon excitation of phonon from the ground state: 0 is the ground state of the crystal, 1 is the phonon energy level, ω_{ph} is the photon energy frequency, ω is the phonon transition frequency.

The Hamiltonian of coherent phonons and photons, homogeneously distributed in crystal, is represented as follows

$$H = \hbar\omega a_k^+ a_k + \hbar\omega_{ph} c_k^+ c_k + i\hbar g_k (c_k^+ + c_{-k})(a_k^+ + a_{-k}) + \frac{\hbar\hat{v}}{2} a_k^+ a_k^+ a_k a_k, \quad (1)$$

where a_k^+ (a_k) and c_k^+ (c_k) are the operators of creation (annihilation) of phonon and photon, respectively, with the wave vector $\vec{k}$ and energies $\hbar\omega$ and $\hbar\omega_{ph}$, g_k is the constant of phonon-photon interaction, $\hat{v}$ is the constant of the elastic phonon-phonon interaction.

In what follows, we set $\hbar = 1$ and go over to amplitude-phase variables

$$\begin{aligned} a_k &= \sqrt{n} e^{-i\varphi_a + ikx}, & a_k^+ &= \sqrt{n} e^{i\varphi_a - ikx}, \\ c_k &= \sqrt{f} e^{-i\varphi_c + ikx}, & c_k^+ &= \sqrt{f} e^{i\varphi_c - ikx}, \end{aligned} \quad (2)$$

where $n, \varphi_a, f, \varphi_e$ are the numbers and phases of phonons and photons.

In these variables, the system Hamiltonian takes the form

$$H = \omega n + \omega_{ph} f + 2g\sqrt{nf}[\sin(\varphi_a - \varphi_e) + \sin(\varphi_a + \varphi_e)] + \frac{\hat{v}}{2} n^2. \quad (3)$$

Taking into account (3), the temporal evolution of coherent phonons and photons is described by the following system of equations

$$\begin{aligned} \frac{dn}{d\tau} &= -\Lambda(nf)^{1/2} (\cos\psi + \varepsilon \cos\varphi), \\ \frac{df}{d\tau} &= \Lambda(nf)^{1/2} (\cos\psi - \varepsilon \cos\varphi), \\ \frac{d\psi}{d\tau} &= \Delta + \frac{\Lambda}{2} \left[\left(\frac{f}{n} \right)^{1/2} + \left(\frac{n}{f} \right)^{1/2} \right] [(\sin\psi + \varepsilon \sin\varphi) + \nu n], \\ \frac{d\varphi}{d\tau} &= 2 + \Delta + \frac{\Lambda}{2} \left[\left(\frac{f}{n} \right)^{1/2} - \left(\frac{n}{f} \right)^{1/2} \right] [(\sin\psi + \varepsilon \sin\varphi) + \nu n], \end{aligned} \quad (4)$$

where

$$\Lambda = \frac{2g}{\omega_{ph}}, \Delta = \frac{\omega - \omega_{ph}}{\omega_{ph}}, \nu = \frac{\hat{v}}{\omega_{ph}}, \tau = \omega_{ph} t, \psi = \varphi_a - \varphi_e, \varphi = \varphi_a + \varphi_e. \quad (5)$$

3. Nonlinear dynamic in resonant approximation. Exact solutions

The parameter ε is introduced in such a manner that ($\varepsilon = 0$) when the non-resonant terms are ignored, and ($\varepsilon = 1$) when non-resonant terms are taken into account.

In the resonance approximation ($\varepsilon = 0$) the system has a motion integral

$$n + f = c. \quad (6)$$

Introducing notations: $\bar{n} = n/c$, $\bar{v} = \nu c$, $\bar{p} = p/c$, going to variables $(\dot{n}, n)$ and dropping the bar, one obtains from (4)

$$\dot{n}^2 = \Lambda^2 n(1-n) - \left(p - \Delta n - \frac{\nu}{2} n^2 \right)^2, \quad (7)$$

where $p = \Delta n + \Lambda n^{1/2} (1-n)^{1/2} \sin\psi + \frac{\nu}{2} n^2$ is an additional motion integral acting as a Hamiltonian in the space of variables (n, ψ)

$$\begin{aligned} \dot{\psi} &= \frac{\partial P}{\partial n} = \Delta + \frac{\Lambda}{2} \left[\left(\frac{1-n}{n} \right)^{1/2} - \left(\frac{n}{1-n} \right)^{1/2} \right] \sin\psi + \nu n, \\ \dot{n} &= -\frac{\partial P}{\partial \psi} = -\Lambda n^{1/2} (1-n)^{1/2} \cos\psi. \end{aligned} \quad (8)$$

From now on, the case $P = 0$ is studied, that corresponds to the initial condition $n_0 = 0$. If the equation

$$\Lambda^2 (1-n) - n \left(\Delta - \frac{\nu}{2} n \right)^2 = 0 \quad (9)$$

has one real root and two complex roots, the solution of (7) is

$$n = \frac{\frac{\cos \theta_1}{\cos \theta_2} \alpha c n^2 \left[\frac{v \tau d}{4 \sqrt{\cos \theta_1 \cdot \cos \theta_2}} + F(\phi_0), K \right]}{\frac{\cos \theta_1}{\cos \theta_2} c n^2 \left[\frac{v \tau d}{4 \sqrt{\cos \theta_1 \cdot \cos \theta_2}} + F(\phi_0), K \right] + s n^2 \left[\frac{v \tau d}{4 \sqrt{\cos \theta_1 \cdot \cos \theta_2}} + F(\phi_0), K \right]}, \quad (10)$$

where α is the real root of equation (9), $b \pm id$ are the complex roots, $\theta_1 = \arctg\left(\frac{\alpha - b}{d}\right)$, $\theta_2 = \arctg\left(-\frac{b}{d}\right)$, $F(\phi_0)$ is the elliptic integral of the first kind,

$K = \cos \frac{\theta_1 - \theta_2}{2}$ is the modulus of the elliptic function, $\theta_0 = 2 \arctg \sqrt{\frac{\cos \theta_1}{\cos \theta_2} \frac{\alpha - n_0}{n_0}}$, n_0 is the

initial concentration of phonons.

If (9) has three different real roots, the solution of (7) has the form

$$n = \frac{n_1 n_3 s n^2 \left[\frac{v \tau}{2} \sqrt{n_2 n_{13}} + F(\phi_0), K \right]}{n_1 - n_3 c n^2 \left[\frac{v \tau}{2} \sqrt{n_2 n_{13}} + F(\phi_0), K \right]}, \quad (11)$$

where $n_3 < n_2 < n_1$ are the roots of equation (9), $F(\phi_0)$ is the elliptic integral of the first kind, $K = \sqrt{n_{12} n_{13} / n_{13} n_2}$ is the modulus of the elliptic function, $\phi_0 = \arcsin \sqrt{n_{13} n_0 / n_3 n_{10}}$, $n_{ik} = n_i - n_k$.

For $\Delta = -2\Lambda$ and $v = 4\Lambda$, equation (9) has a single real root and a double real root. The phase trajectory of the system is separatrix. The solution on separatrix assumes the form:

$$n = \frac{n_1 n_2 s h^2 \left[\frac{v \tau}{2} \sqrt{n_2 n_{12}} + \ln \left(\operatorname{tg} \frac{\phi_0}{2} + \frac{\pi}{4} \right) \right]}{n_1 c h^2 \left[\frac{v \tau}{2} \sqrt{n_2 n_{12}} + \ln \left(\operatorname{tg} \frac{\phi_0}{2} + \frac{\pi}{4} \right) \right] - n_2}; \quad (12)$$

that is, all photons are converted into phonons, that finishes the system evolution. Here $n_1 = 1, n_2 = 1/2$.

Various solutions of equation (7) are determined by the shape of the potential curve (Fig. 2).

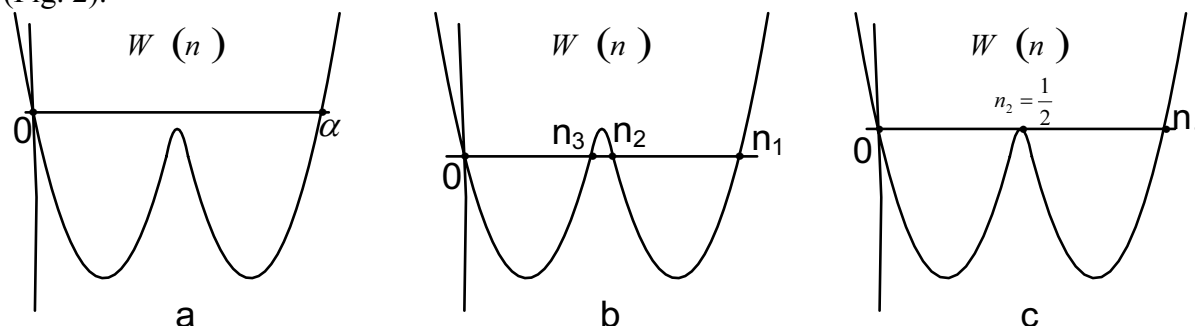


Fig. 2. Potential energy $W(n)$ of the nonlinear oscillation when equation (9) has one real and two complex roots (a), three different real roots (b), one real roots and one real double root (c). Horizontal line corresponds to the total energy of the oscillator.

4. Nonlinear dynamics with account of non-resonant terms. Computer simulation

The variation of the motion integral P under the action of perturbation, caused by non-resonant terms of the Hamiltonian, is described by the equation

$$\frac{dp}{d\tau} = \frac{\frac{\Lambda^2}{2} \left(2n-1 \sin(\psi + \varphi) + \Lambda \left[n(c-n)^{1/2} \right] (\Delta + \nu n) \cos \varphi \right)}{1 + \left(p - \Delta n - \frac{\nu n^2}{2} \right) / 2(1-n)}. \quad (13)$$

In the separatrix vicinity one can go over to discrete transformation

$$\begin{aligned} P_{m+1} &= P_m + \bar{\Delta}P, \\ \Theta_{m+1} &= \Theta_m + \frac{4\pi}{\omega(P_{m+1})} = \Theta_m + \frac{4\pi}{\omega(P_m)} - \frac{4\pi}{\omega^2(P_{m+1})} \frac{d\omega(P_m)}{dP_m} \bar{\Delta}P, \\ \bar{\Delta}P &= \frac{\Lambda}{2} \int A(\tau) \sin \Theta d\tau, \end{aligned} \quad (14)$$

where $\frac{d\tilde{\theta}}{d\tau} = 2 + 2\Delta + O(\lambda)$ and $A(\tau)$ is an elliptic function with period $2\pi/\omega(p)$,

height ~ 1 and width $2\pi/\omega_0$, ω_0 is the frequency of small oscillations of the system.

The nature of the solution of equation (16) is determined by the quantity M

$$M = \frac{4\pi}{\omega^2(P)} \left| \frac{d\omega(P)}{dP} \right| \bar{\Delta}P. \quad (15)$$

When $M \ll 1$, the system performs quasi-periodic oscillations. If $M \gg 1$, the motion becomes stochastic, the stochastic layer being situated within the interval

$$0 < |P| < 16\sqrt{2}\pi\nu \exp(-4\pi \frac{\alpha}{\gamma}), \quad (16)$$

where constant α is on the order of unity.

The stochastic region widens while ν increases and at certain values of ν extends over the entire phase space region.

Generally, equation system (4) has one motion integral — the energy of the system — and the motion region in the phase space is a three-dimensional hypersurface, defined by (1) in a four-dimensional phase space. When non-resonant terms are neglected ($\varepsilon = 0$), an additional motion integral P appears. In that case, temporal evolution of coherent quasiparticles represents periodical non-linear oscillation, described by equations (10) and (11), or aperiodical oscillations, described by (12). As the separatrix is approached, the period of oscillation increases and becomes infinite on separatrix.

When the non-resonant terms are taken into account, the breakdown of motion integral P occurs. The motion becomes quasiperiodical. In the phase space, the motion is represented by a trajectory that envelopes onto tore. The main amplitude caused by resonant terms is modulated by the sub-harmonics generated by non-resonant terms (Fig. 3).

Investigation of system motion in the separatrix vicinity is of a particular interest. The account of non-resonant terms of the interaction Hamiltonian leads to disappearance of aperiodical oscillation mode. The phonon number varies quickly from 0 to $1/2$, where the velocity of phonon number variation decreases sharply. Phonon number oscillates around the value of

1/2 in a certain time interval, due to nonresonant terms, and, after crossing the point of potential energy maximum, increases up to 1. Then the phonon number continues its evolution in the opposite direction. So, the non-resonant terms destroy the mode of aperiodical motion, allow the system to pass over the potential barrier and to move from a potential hole into another one (Fig. 4).

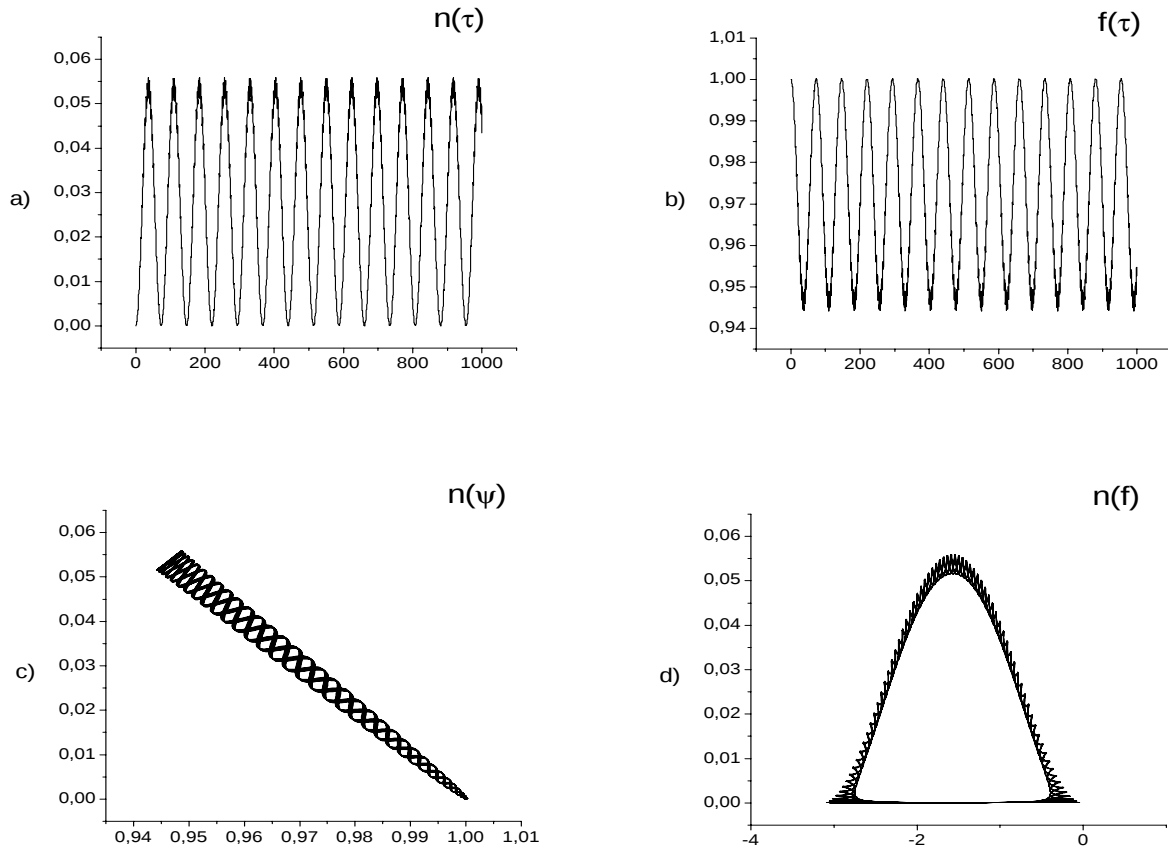


Fig. 3. Temporal evolution of the numbers of phonons (a) and photons (b), projection of the phase trajectory on the planes coherent phase-phonon number (c), photon number - phonon number (d) when $\Lambda = 2 \cdot 10^{-2}$, $\Delta = 8 \cdot 10^{-2}$, $\nu = 8 \cdot 10^{-2}$, $n_0 = 10^{-8}$, $f_0 = 1$, $\varphi_{a0} = \varphi_{e0} = 10^{-8}$.

Let us study the influence of the variation of the parameters Λ , Δ , and ν upon the system behavior in the separatrix vicinity. For constant values of Λ and Δ , when ν decreases in an interval $\frac{\Delta\nu}{\nu} \sim 2,5 \cdot 10^{-3}$ in the separatrix vicinity, the phonon numbers varies approximately from 0 through 1, the system passing over the maximum of potential curve in the point 1/2. Then the oscillation amplitude of phonon number decreases sharply from 1 to 1/2, that denotes the system is unable to pass over the potential barrier. In Fig. 5, the phonon number evolution is shown for parameter values $\Lambda = 5 \cdot 10^{-3}$, $\Delta = -10^{-2}$, $\nu = 1,995 \cdot 10^{-2}$ (a) and $\nu = 1,994 \cdot 10^{-2}$ (b), respectively. Further, the oscillation amplitude decreases and the oscillation frequency increases.

The increase of the value ν in the separatrix vicinity leads to gradual decrease of both the amplitude and the oscillation period.

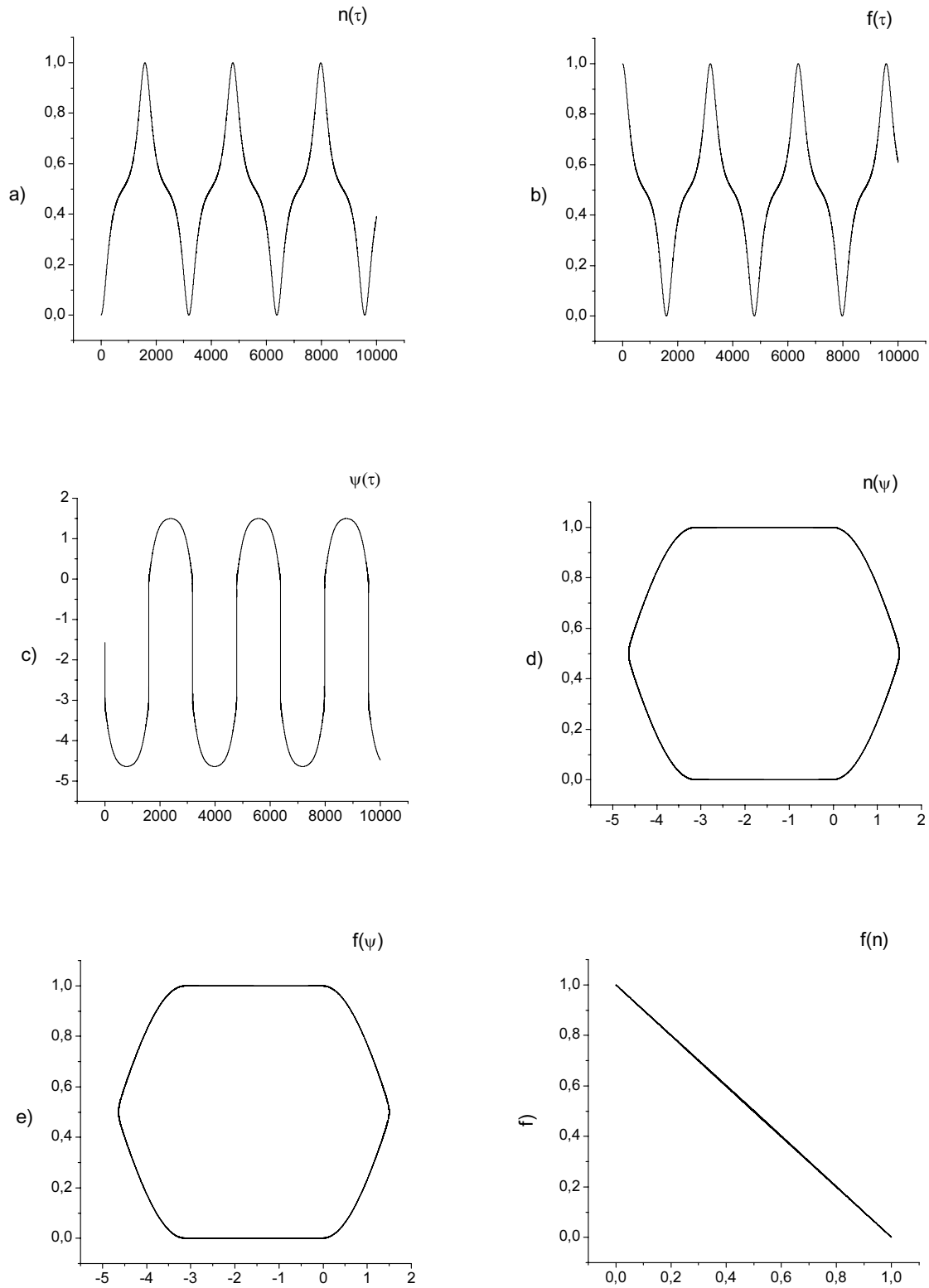


Fig. 4. Temporal evolution of the numbers of phonons (a), phonons (b), resonant phase (c), and the projection of the phase trajectory on the planes coherent phase-phonon number (d), coherent phase – photon number (e), photon number - phonon number (f) when $\Lambda = 5 \cdot 10^{-3}$, $\Delta = -10^{-2}$, $\nu = 2 \cdot 10^{-2}$, $n_0 = 10^{-8}$, $f_0 = 1$, $\varphi_{a0} = \varphi_{e0} = 10^{-8}$.

Fixing Λ and ν , the increase in the Δ value was observed to lead to slow decrease in both the amplitude and the oscillation period, but when the Δ value is diminished, a skip of phonon number amplitude from 1 to $1/2$ occurs (Fig. 6).

Fixing Δ and ν , at the diminution of the Λ value in the separatrix vicinity, a skip of the phonon number amplitude occurs from 1 to $1/2$, while the increase in Λ leads to the increment of oscillation frequency without significant influence on the oscillation amplitude (Fig. 7).

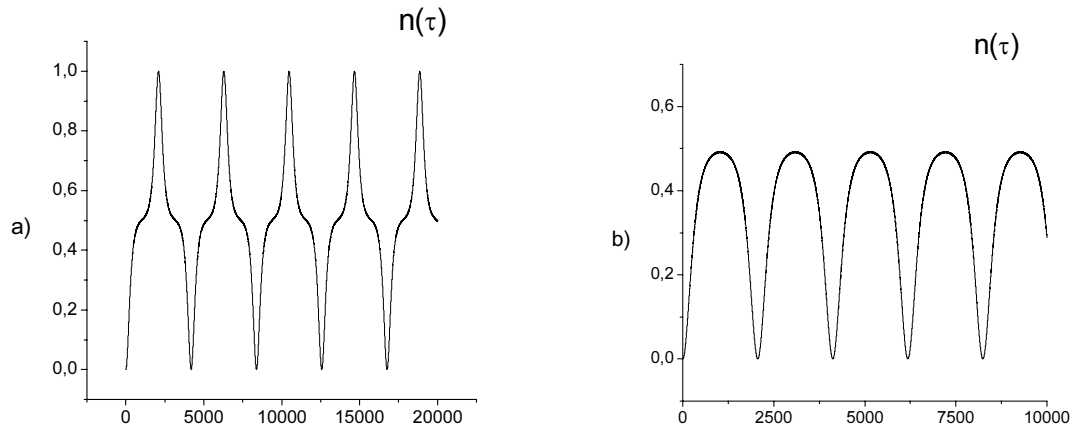


Fig. 5. Temporal evolution of phonon number when $\Lambda = 5 \cdot 10^{-3}$, $\Delta = -10^{-2}$, $\nu = 1,995 \cdot 10^{-2}$ (a) and $\nu = 1,994 \cdot 10^{-2}$ (b), $n_0 = 10^{-8}$, $f_0 = 1$, $\varphi_{a0} = \varphi_{e0} = 10^{-8}$.

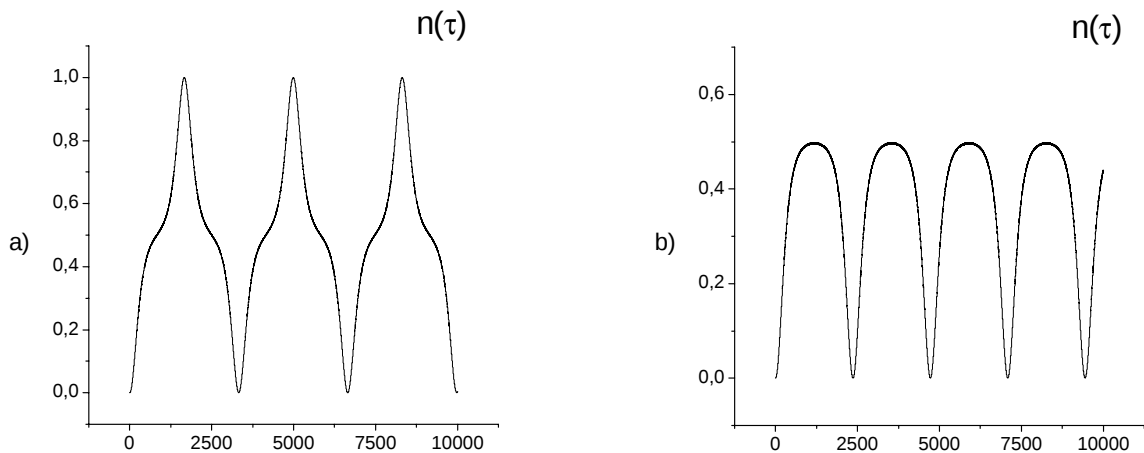


Fig. 6. Temporal evolution of phonon number when $\Lambda = 5 \cdot 10^{-3}$, $\nu = 2 \cdot 10^{-2}$, $\Delta = -1,001 \cdot 10^{-2}$ (a) and $\Delta = -1,002 \cdot 10^{-2}$ (b), $n_0 = 10^{-8}$, $f_0 = 1$, $\varphi_{a0} = \varphi_{e0} = 10^{-8}$.

Thus, the increase in either ν , Δ , or Λ values in the separatrix vicinity diminishes the potential energy value in the point $1/2$ under the system total energy value, while the decrease in ν , Δ or Λ values leads to the increment of potential energy over the total energy of the system at point $1/2$.

The amplitude of resonant oscillation is determined by the width of the potential hole of the system, since the non-resonant mode amplitude is determined by Λ .

The amplitude of non-resonant modes can achieve the magnitude of resonant oscillations in the separatrix vicinity, if Λ is increased up to 1 and Δ and ν are increased proportionally to fit the relations $\Delta = -2\Lambda$ and $\nu = 4\Lambda$. But the maintenance of the system in the separatrix vicinity by increasing in Λ can lead into the region of unread values, such as $\Delta \leq -1$.

Numeric simulation shows that increase in Λ up to 1 both in the separatrix vicinity and far from it leads to an increment of the torus diameter and deformation of torus, but the system dynamics remains quasiperiodical and the power spectrum remains discrete.

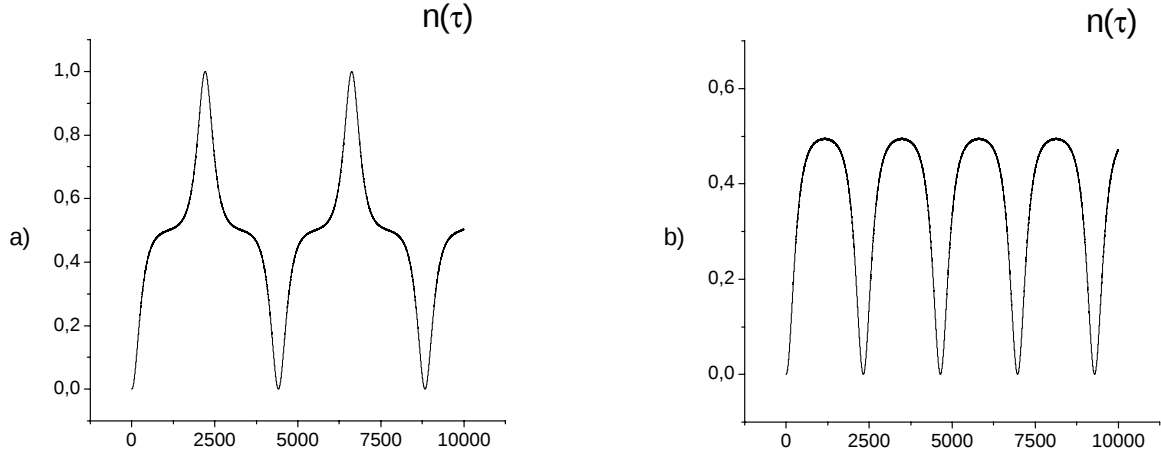
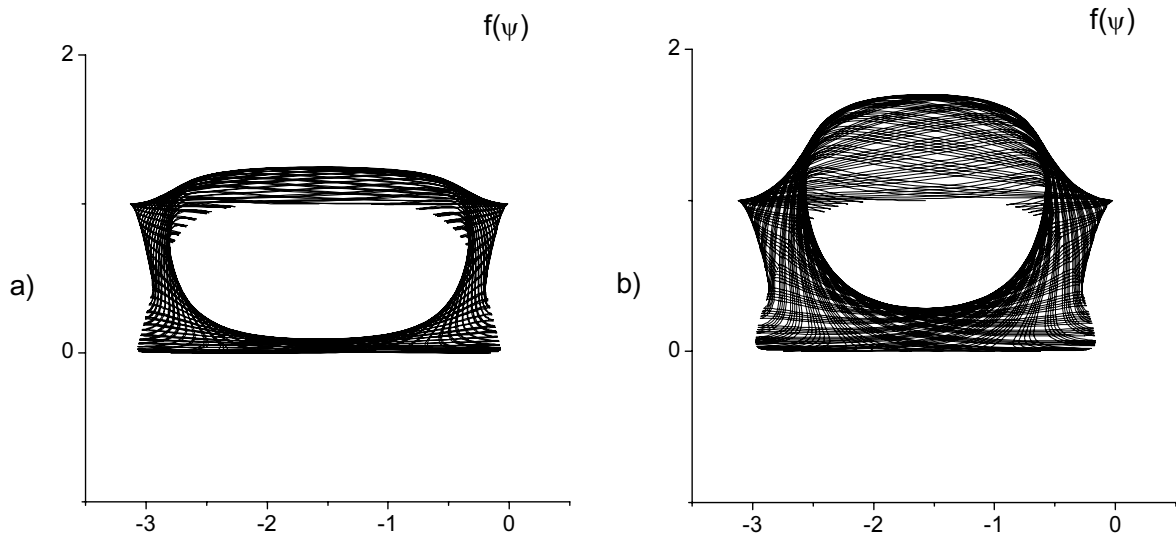


Fig. 7. Temporal evolution of phonon number when $\Delta = -10^{-2}$, $\nu = 2 \cdot 10^{-2}$, $\Lambda = 4,987 \cdot 10^{-3}$ (a) and $\Lambda = 4,986 \cdot 10^{-3}$ (b), $n_0 = 10^{-8}$, $f_0 = 1$, $\varphi_{a0} = \varphi_{e0} = 10^{-8}$.

In Fig. 8, the projection of the phase trajectory of the system on the $f(\psi)$ plane is represented for different values of Λ . The diameter of the perturbed torus is shown to increase while Λ increases.

In Fig. 9, the power spectra of the phonon number oscillation for various values of Λ are represented.

The power spectrum is observed to have a finite number of lines.



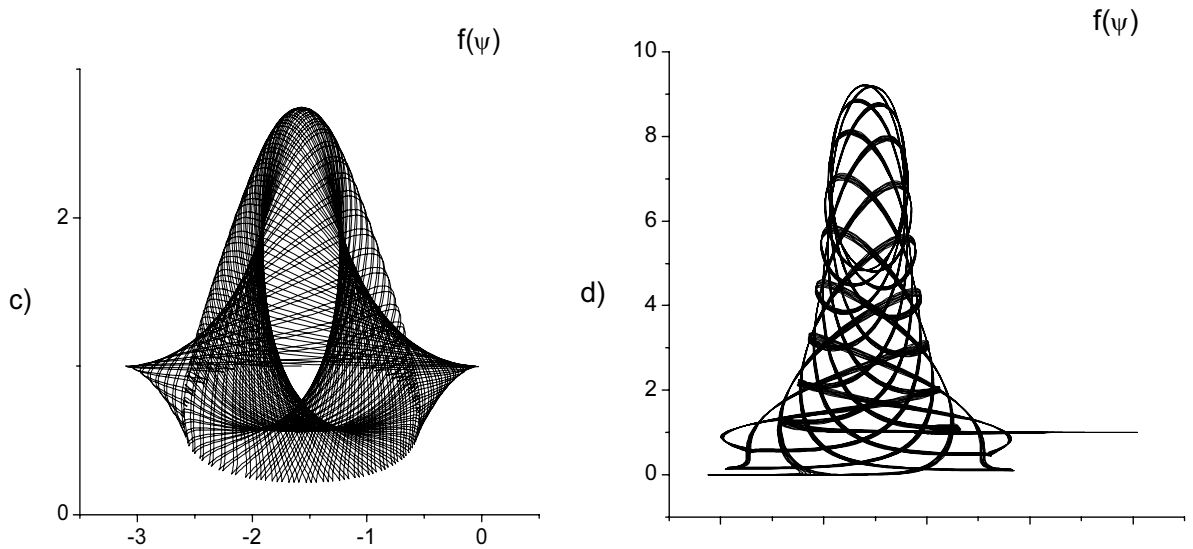


Fig. 8. Projection of phase trajectory on plane coherent phase - phonon number at $n_0 = 10^{-8}$, $f_0 = 1$, $\psi_{n0} = 10^{-8}$, $\psi_{e0} = 1.571$, $\Delta = 10^{-2}$, $\nu = 2 \cdot 10^{-2}$ $\Lambda = 0,5$ (a), $\Lambda = 0,7$ (b), $\Lambda = 0,8$ (c) and $\Lambda = 1$ (d).

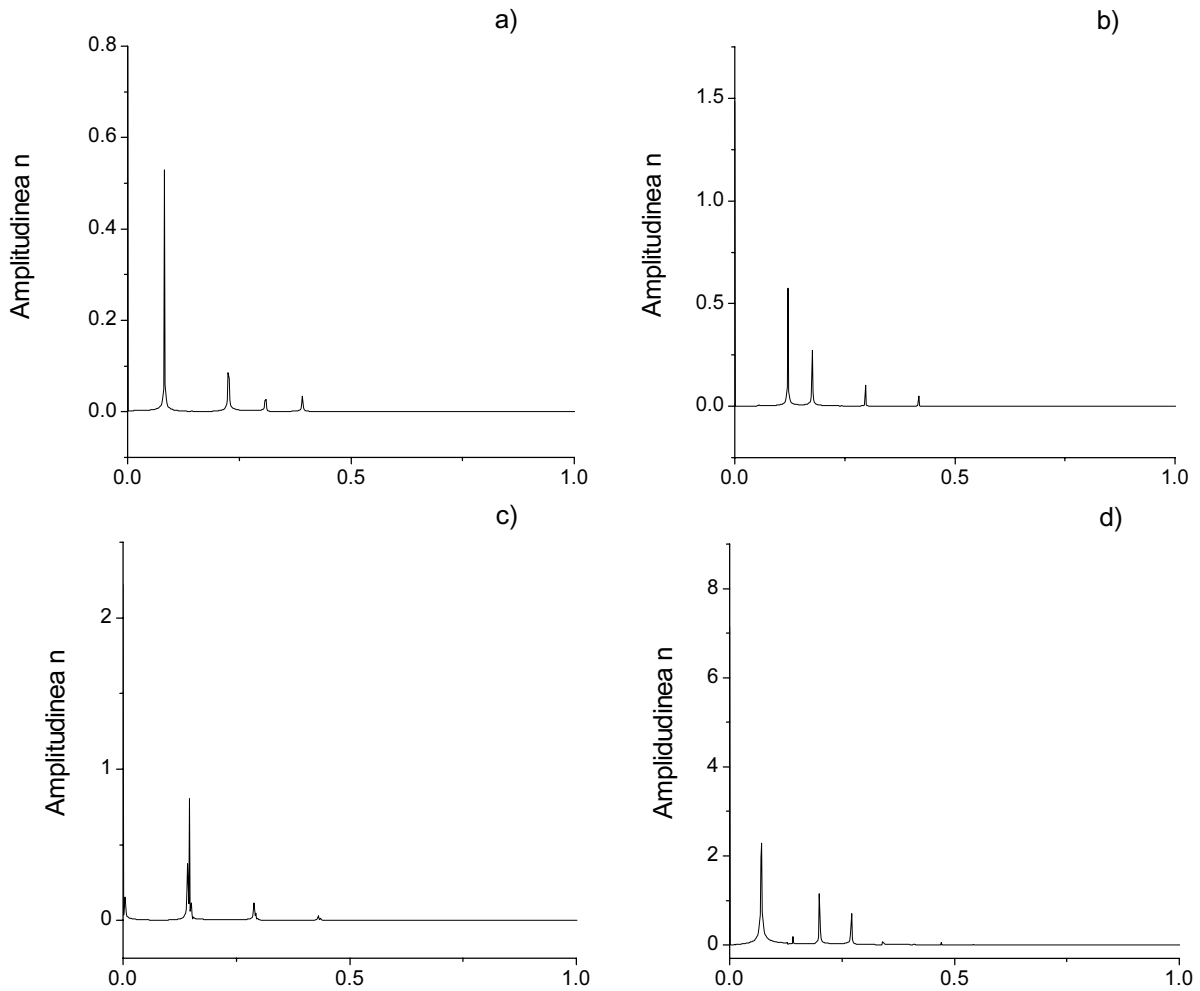


Fig. 9. Power spectrum of the oscillation of phonon number at $n_0 = 10^{-8}$, $f_0 = 1$, $\psi_{n0} = 10^{-8}$, $\psi_{e0} = 1.571$, $\Delta = 10^{-2}$, $\nu = 2 \cdot 10^{-2}$ $\Lambda = 0,5$ (a), $\Lambda = 0,7$ (b), $\Lambda = 0,8$ (c) and $\Lambda = 1$ (d).

Another way to achieve values of non-resonant modes of the same magnitudes as of the resonant oscillation ones is to choose the Λ , Δ , and ν parameter values to reduce the width of the potential hole, and, implicitly, the magnitude of resonant oscillations. For a fixed value of Λ and small dimensions of potential hole, achieved by modifying Δ and ν , the superposition of resonant and nonresonant modes leads to doubling and tripling of the system oscillation period.

5. Conclusions

So we can conclude that, due to the fact that nonresonant terms are linear relative to quasiparticle concentrations, the effective dipole momentum does not rise while quasiparticle concentration increases and the stochastic layer in the separatrix vicinity remains very narrow limiting the development of dynamic chaos.

In this paper, the theory of nonlinear dynamics of Bose-condensed was developed, in the Bogoliubov's terms of photons and phonon in biological media. In the resonant approximations, the exact solutions were obtained of the system of nonlinear differential equations that describes the evolution of the Bose-condensed quasiparticles by means of elliptic function. In the nonresonant case, the equation of Hamiltonian variation in the phase space was obtained, and the motion was studied in the separatrix vicinity.

The expression of stochastic layer width was deduced. The computer simulation was carried out, and the temporal evolution of Bose-condensed quasiparticles was obtained.

The stochastic layer was proven to remain narrow and not to extent over the entire phase space at any values of the system parameters that limit the dynamic chaos appearance. The appearance of dynamic chaos requires both the resonant and non-resonant terms to be nonlinear.

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