

EXACT NON-STATIONARY SOLUTION FOR THE DENSITY MATRIX OF SINGLE TWO-LEVEL ATOM

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Abstract

The master equation for short- and long-time interaction intervals of single two-level atom with electromagnetic field vacuum for studying non-exponential emission is proposed. The exact non-stationary solution for the density matrix is obtained.

1. Introduction

The evolution law in quantum mechanics is governed by unitary operators [1]. According with general mathematical properties of unitary operators, the decay of an unstable quantum system cannot be purely exponential. In general, a rigorous analysis based on the Schrodinger equation shows that the decay law is quadratic for very short times [2-5] and governed by a power law for very long times [6-11]. These features of the quantum evolution are well known and discussed in textbooks of quantum mechanics [12-13] and quantum field theory [14]. The temporal behaviour of quantum systems is reviewed in Ref. [15].

In this paper, the density matrix method for study of the initial stage of the evolution problem is proposed and solved analytically. Following the method proposed in paper [16], the master equation for density matrix of spontaneous emission of single atom was obtained. This master equation is similar to density matrix equation of two-level atom in the thermal field, but the Einstein coefficients are explicitly time-dependent at the initial stage. According to the Briegel-Englert method [17], we derived three time-dependent eigenvalues of the non-stationary Liouvillian density matrix. The fourth eigenvalue, together with time-dependent expansion coefficients, is found from the normalized theory of density matrix.

2. Non-stationary master equation with time-dependent coefficients

Let us consider the operator $O(t)$ of an atomic subsystem. It depends on the atomic operators $O(t) = O(R^+(t), R^-(t), R_z(t))$. The Heisenberg equation for the mean value of this operator can be written as

$$\begin{aligned} \frac{d\langle O(t) \rangle}{dt} = & i\omega_0 \langle [R_z(t), O(t)] \rangle + \frac{i}{\hbar} \sum_k g_k \left\{ \langle a_k^+(t) [R^-(t) O(t)] \phi_{12}(k) \rangle \right. \\ & + \langle a_k^+(t) [R^+(t) O(t)] \phi_{21}(-k) \rangle + \langle \phi_{12}(-k) [R^-(t) O(t)] a_k(t) \rangle \\ & \left. + \langle \phi_{21}(k) [R^+(t) O(t)] a_k(t) \rangle \right\}. \end{aligned} \quad (1)$$

The solution of the Heisenberg equation for the electromagnetic field operators a_k^+ and a_k is

$$a_k^+(t) = a_k^+(0) \exp(i\omega_k t) + \frac{ig_k}{\hbar} \int_0^t d\tau \left\{ \varphi_{12}(-k) R^-(t-\tau) + \varphi_{21}(k) R^+(t-\tau) \right\} \exp(i\omega_k \tau),$$

$$a_k(t) = [a_k^+(t)]^+.$$
(2)

After substituting (2) into (1) and averaging over the initial state $\psi(t=0) = |f\rangle|A\rangle$ of the system we exclude the free solution of Eq. (2), $a_k(0)|f\rangle = 0$ and $\langle f|a_k^+(0) = 0$ ($|f\rangle$ and $|A\rangle$ are the states of the electromagnetic field vacuum and of the atom at $t=0$). Thus, Eq. (1) takes the form

$$\frac{d\langle O(t) \rangle}{dt} = i\omega_0 \langle [R_z(t) O(t)] \rangle - \sum_k \frac{g_k^2}{\hbar^2} \int_0^t d\tau \left\{ \left(\varphi_{12}(-k) R^-(t-\tau) + \varphi_{21}(k) R^+(t-\tau) \right) \right. \\ \left. \times \left([R^-(t), O(t)] \varphi_{12}(k) + \varphi_{21}(-k) [R^+(t), O(t)] \right) \right\} \exp(i\omega_k \tau) + h.c.$$
(3)

Following the method of integration of coefficients in the right-hand side of Eq. (3) proposed in papers [16, 18], we obtained the master equation for the dual operator $O(t)$

$$\frac{dO(t)}{dt} = -\frac{(A+B(t))}{2} \{ R^+ R^- O + O R^+ R^- - 2R^+ O R^- \} \\ - \frac{B(t)}{2} \{ R^- R^+ O + O R^- R^+ - 2R^- O R^+ \}.$$
(4)

According to paper [17], from this equation, we obtain the master equation for density matrix $\rho(t)$ of the reduced system

$$\frac{d\rho(t)}{dt} = -\frac{(A+B(t))}{2} \{ R^+ R^- \rho + \rho R^+ R^- - 2R^+ \rho R^- \} \\ - \frac{B(t)}{2} \{ R^- R^+ \rho + \rho R^- R^+ - 2R^- \rho R^+ \}.$$
(5)

Let us introduce the dual Liouvillian $\tilde{L}_R$ in master equations (5)

$$\tilde{L}_R \rho = \frac{(A+B(t))}{2} \{ [R^-, \rho R^+] + [R^- \rho, R^+] \} \\ + \frac{B(t)}{2} \{ [R^+, \rho R^-] + [R^+ \rho, R^-] \},$$
(6)

where $R^+ = |e\rangle\langle g|$, $R^- = |g\rangle\langle e|$, $R_z = \frac{1}{2}(|e\rangle\langle e| - |g\rangle\langle g|)$.

For Einstein coefficients A and $B(t)$ for time intervals, $t \gg 2a/3c$, and neglecting the Lamb shift (here a is the Bohr radius), we use the formulas [16, 18]

$$A = \frac{1}{\tau_0}, \quad B(t) = \frac{1}{\pi\tau_0} \left(Si(\omega_0 t) - \pi/2 + \frac{\cos(\omega_0 t)}{\omega_0 t} \right),$$
(7)

Let us try to generalize the Briegel-Englert method [17] for non-stationary case. From quantum mechanics it is well known that in some non-stationary formal problems the Schrödinger equation can be solved for eigenfunctions and eigenvalues.

3. The solution of non-stationary master equation and its analysis

Following the method [17] for non-stationary Lindblad operator [19] $L_R(t)$, we find a non-stationary eigenvalue

$$L_R(t)\rho_n = \lambda_n(t)\rho_n.$$
(8)

So, the density matrix will be found in the following expansion representation

$$\rho(t) = \sum_n c_n(t) \exp \left\{ \int_0^t \lambda_n(\tau) d\tau \right\} \hat{\rho}_n, \quad (9)$$

where $c_n(t)$ are time-dependent matrix coefficients. This solution (9) satisfies non-stationary master equation (5). Returning to Eq. (6), we can observe that the following negative eigenvalues exist for eigenstates of operators R_z , $R_{\pm}$

$$\begin{aligned} L_R R_z &= -(A + 2B(t)) R_z, \\ L_R R^{\pm} &= -\frac{1}{2} (A + 2B(t)) R^{\pm} \end{aligned} \quad (10)$$

Then the linear combination of their solutions will be a solution too. The difficulty arises with the matrix I , but if we assume $L_R I = 0$, then it is necessary to put a condition for the time-dependent coefficients of matrix R_z and unitary matrix I . Using expansion (9) generalized for the explicitly time-dependent coefficients in non-stationary part of Liouville equation (5), we can write the non-stationary solution with time-dependent coefficients in the following form

$$\begin{aligned} \rho(t) &= f(t) I + c_2(t) R_z \exp \left\{ - \int_0^t [A + 2B(\tau)] d\tau \right\} \\ &+ (c_3 R^+ + c_4 R^-) \exp \left\{ - \frac{1}{2} \int_0^t [A + 2B(\tau)] d\tau \right\}. \end{aligned} \quad (11)$$

Indeed, after using this solution in Eq. (5) and taking into consideration the normalization condition $Sp\{\rho(t)\} = 1$, we find the coefficient $f(t) = 1/2$. After that we must require the following coefficient for the non-stationary coefficients $c_2(t)$

$$c_2(t) = \left\{ 1 - A \int_0^t dt' \exp \left\{ \int_0^{t'} [A + 2B(\tau)] d\tau \right\} \right\}. \quad (12)$$

Thus, we have obtained the following particular non-stationary solution $\rho_1(t)$

$$\rho_1(t) = R_z \left\{ 1 - A \int_0^t dt' \exp \left\{ \int_0^{t'} [A + 2B(\tau)] d\tau \right\} \right\} \exp \left\{ - \int_0^t [A + 2B(\tau)] d\tau \right\} + 1/2. \quad (13)$$

The constants c_3 and c_4 in Eq. (11) were found from the initial conditions, $\rho|_{t=0}$

$$c_3 = \rho_{21}, \quad c_4 = \rho_{12} \quad (14)$$

The non-stationary solution with time-dependent coefficients for a two-level atom interacting with one mode of the quantized photon field reads

$$\begin{aligned} \rho(t) &= 1/2 + R_z \left\{ 1 - A \int_0^t dt' \exp \left\{ \int_0^{t'} [A + 2B(\tau)] d\tau \right\} \right\} \exp \left\{ - \int_0^t [A + 2B(\tau)] d\tau \right\} \\ &+ (\rho_{21} R^+ + \rho_{12} R^-) \exp \left\{ - \frac{1}{2} \int_0^t [A + 2B(\tau)] d\tau \right\}, \end{aligned} \quad (15)$$

which satisfies equation (5). It is not difficult to observe that $\frac{d}{dt} Sp(\rho(t) R_z(0))$ coincides with equation for this mean value proposed in papers [16, 18],

$$\frac{d\langle R_z(t) \rangle}{dt} = -[A_{II}(t) + B(t)] [\langle R_z(t) \rangle + 1/2] + B(t) [1/2 - \langle R_z(t) \rangle]. \quad (16)$$

Using solution (15) we can find the mean value of R_z : ($Sp(\rho(t)R_z(0))$). In order to get this time-dependent function we have to integrate the exponent

$$\mathcal{I}(t') = \int_0^{t'} [A + 2B(\tau)] d\tau, \quad (17)$$

which leads to the following expression

$$\begin{aligned} \mathcal{I}(t') = At' + \frac{2}{\pi\tau_0} \left[t' Si(\omega_0 t') + \frac{1}{\omega_0} (\cos(\omega_0 t') - 1) - \frac{\pi}{2} t' \right. \\ \left. + \frac{1}{\omega_0} \{Ci(\omega_0 t') - Ci(\omega_0 a/c)\} \right]. \end{aligned} \quad (18)$$

Finally, introducing this result into Eq. (15), we obtain the solution with non-stationary time-dependent coefficients for the density matrix $\rho(t)$

$$\begin{aligned} \rho(t) = 1/2 + R_z \left\{ 1 - A \int_0^t dt' \exp\{\mathcal{I}(t')\} \right\} \exp\{-\mathcal{I}(t)\} \\ + (\rho_{21} R^+ + \rho_{12} R^-) \exp\left\{-\frac{1}{2} \mathcal{I}(t)\right\}. \end{aligned} \quad (19)$$

The non-diagonal matrix elements are different from zero when the atom, in the initial stage, was in the inseparable state (the wave function $\psi = \alpha|g\rangle + \beta|e\rangle$)

$$\rho(t=0) = |\psi\rangle\langle\psi| = \alpha^2 |g\rangle\langle g| + \beta^2 |e\rangle\langle e| + \alpha\beta^* |g\rangle\langle e| + \beta\alpha^* |e\rangle\langle g|. \quad (20)$$

From (20) we obtain

$$\rho_{12} = \alpha\beta^*, \quad \rho_{21} = \beta\alpha^*, \quad \rho_{11} = |\alpha|^2, \quad \rho_{22} = |\beta|^2. \quad (21)$$

The exact solution of every atomic operator can be found taking into account Eq. (19). For example, for atomic inversion $\langle R_z(t) \rangle$ we get

$$\begin{aligned} \langle R_z(t) \rangle = Sp\{\rho(t)R_z(0)\} \\ = \frac{1}{2} \left\{ 1 - A \int_0^t dt' \exp(\mathcal{I}(t')) \right\} \exp(-\mathcal{I}(t)). \end{aligned} \quad (22)$$

In other words, the exact solution gives us the possibility to study all quantum properties of the evolution of the excited atom during spontaneous emission process.

4. Conclusions

In this paper, a new non-stationary method has been developed for studying non-exponential emission. In this method the Liouvillian equation coefficients are time-dependent. So, it can be applied to non-exponential spontaneous emission of one or more radiators. In general, the method proposed in this paper is useful and convenient when we investigate the quantum properties of the systems under study.

In summary, we have presented the master equation solution in which the linear Lindblad operator $\mathcal{L}_R$ is expressed in terms of the time-dependent coefficients. Using the method of eigenvalues and eigenfunctions for non-stationary case, we have exactly solved the linear Lindblad master equation with non-stationary Einstein coefficients. Applying these formulas, we have found the time dependence of inversion during the spontaneous emission process. This model is often encountered in quantum optics, quantum measurement, and quantum information processing.

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